Mathematical Archaeology on Pupils' Mathematical Texts

Un-earthing of Mathematical Structures

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ABSTRACT

The main focus for this study is un-earthing mathematical structures in pupils' mathematical texts. This un-earthing process can be characterised as mathematical archaeology. The origin of these texts date back to the early 1990's when the Norwegian journal for mathematics education Tangenten was founded by Professor Stieg Mellin-Olsen. It was decided that each issue of the journal should contain a competition section for Scandinavian school-classes in the compulsory school; i.e. from grade 1 to grade 9. The focus for the competition was the solving of open-ended tasks by the pupils in collaboration. All together eight investigational tasks were given before the competition came to an end. The data basis for this thesis is the pupils' solutions for the first six of these tasks, which constitute of 23 mathematical texts. One premise for a class to participate in the competition was that the teacher had to play a passive role, which meant that intervening in the solving process with hints and advisees was not allowed. The study is falling within the naturalistic inquiry paradigm.

The purpose of the analysis method applied in this study is the explicit-making of the mathematical structures inherent in the pupils' solutions. The only concern for the analysis is the uncovering of mathematical structures and not possible arithmetical errors or inferred cognitive processes. Since the interest is in analysing the produced solutions in terms of mathematical structures the process can be viewed as a decoding process. A brief description of this method could be 'analysing the pupils' solutions through the glasses of a mathematician'. The method is thus one of decoding the solutions which is manifested in the learners' language into the formal mathematical language. The term 'mathematical language' means in this context a formal mathematical language; a language that uses mathematical symbols/notations. For some of the solutions or parts of a solution, this decoding process was relatively simple; i.e. the mathematical meaning of the text was clear, and a direct decoding was possible. In some cases a direct decoding was not straight-forward, the decoding process gave a product with gaps, and the decoding process lead to an incomplete mathematical description. In this case it was both necessary and required to 'fill the gaps' in order to get a solution written in a mathematical language. In doing so an emendation process was entered.

The first research question focused on the mathematical structures inherent in the solution procedures applied by the pupils in the analysed texts. This analysis revealed that the pupils in their solving procedures heavily relied on some fundamental mathematical structures. In each answer one or more relationship was stated, some of these relationships had an explicit structure, while others had a recursive structure. Mathematically the stated relationships can be identified as functions, in some instances functions with more than one independent variable.

The mathematical structures inherent in the solution procedures can be localised at two levels, a top level and an underlying level. The top level embraces the mathematical structures that can be identified in the relationships or rules that the pupils clearly stated in their answers. At this level the mathematical structures are clearly visible. The underlying level consists of mathematical structures that are applied in the solving procedures but not emphasised or (directly) focused by the pupils in their stated relationships or rules. The mathematical structures identified in the underlying level are to a certain extent hidden or concealed. The function structure is visible at the top level, but it is also present in the underlying level. At the top level functions are identified directly as tables or through the rules or relationships stated as a formula presented in a mathematical symbolic language or in the Norwegian (Swedish) language. The function structure at the underlying level can be identified (indirectly) through the solving procedure. Another underlying mathematical structure that has been identified in some of the solution procedures is equivalence relations. A third mathematical structure identified in the solution procedures is the use of figurative numbers.

The pupils' solutions were produced in collaboration in their ordinary class context. A crucial question in that respect is factors influencing the solving process and hence the solving procedures. Four factors are discussed; the formulation of the task, the mathematical maturity to the pupils, the influence of the teacher and the competition rules. It is concluded that two of these four factors play a more prominent role related to the influencing question. These factors are the teacher and the competition rules. The teacher's influence is indirectly as responsible for the organising of the solution process, the competition rules through the collaborative aspect. It seems that the formulation of the tasks are the factor that is the least important of the four factor discussed. The influence of the factor linked to the mathematical maturity to the pupils is problematic. There are indications that this factor is decisive for the out come of the solving process, but there is also indication of the opposite, consequently it is questionable to draw any conclusion concerning the influence of this factor.

The second research question focused on mathematical archaeology and discusses if it is a suitable tool for increasing the knowledge about pupils' mathematical activity and their mathematical thinking. It is established that mathematical archaeology applied on pupils' solutions of open-ended tasks is a suitable tool, and that explicit-making of the mathematics in the texts give valuable information of the pupils' mathematical activity and their mathematical thinking. The archaeology carried through in this study has revealed that the pupils in their collaborative solutions in some cases have used mathematical structures not taught in school. Most likely, the pupils will first be taught some of these mathematical structures in mathematics courses at university or college level. It is concluded that the explicit-making of the mathematics identified in pupils' mathematical texts has implications in relation to both a local and a global perspective. The local perspective is related to the practice at the classroom level, and concerns the learning process in the individual classroom. It has implications for the teaching of the subject; i.e. how to plan and to carry through the teaching, and how to revise and adjust the teaching. This arena is the arena for mathematics teachers, but it is also a field where researchers in mathematics education should play important roles. These roles should be played in close collaboration with the mathematics teachers. The global perspective concerns the school at large, such as development of curricula, the design and development of suitable tasks and teaching elements, education of mathematics teachers, etc. This arena, the global level, is the field for the researcher in mathematics education. However, the roles played by the actors linked to these two levels are not focused in this study.

PRELIMINARY PART

A PERSONAL PEDAGOGICAL STORY

This section gives a short description of the pedagogical development to the researcher in the period ahead of the start of this study, which means that the section is not directly related to the study, but acts as a prologue for the study. The significance of the section is that the result of this development has had great influence of the pedagogical basis on which this study rests. Pedagogical basis is here used in a broad setting. It includes the conception on the nature of mathematics which in turn have an impact on how to teach mathematics, and the learning process in this subject. As a consequence of that influence it was decided to elucidate this pedagogical platform.

The description in this section is mainly based on memory, reflection on some events, text-books used, written notes for the lessons and problems constructed for written examinations.

Introduction

The first half of the 1970's was in Norway, as in many countries, a period where the school mathematics was focused. A temporary national curriculum for compulsory education in Norway was published in 1971 ((The) Ministry of Education and Research, 1971). The mathematics curriculum in this national curriculum caused discussions and reactions, mainly because the curriculum to a large extend was influenced by the 'New Math' movement. With respect to the mathematical curriculum in the Norwegian compulsory school, the first half of the 1970's was in many ways, a turbulent period of time. For details see Gjone (1985a, 1985b). It was in that turbulent period, August 1971, this researcher started his career as a mathematics teacher. The first position held was as a teacher at a lower secondary school. In August 1971 the mathematical background for this teacher was *hovedfag* (an extended master degree) in pure mathematics. The pedagogical background was one semester at a pedagogical seminar, and no teaching experience beyond the experience at the pedagogical seminar.

Teaching at a lower secondary school

The teaching style at the lower secondary school was what may be called traditional. Most of the lessons could be divided in three different and often distinct parts. The first part was control of the pupils homework which was exercises or problems from the pupils text-books.

The second part was to cover the next pages in the text-book, and to prepare for the pupils' homework. In the last part of the lesson the pupils were working on their own on exercises.

The mathematical text-books, Christoffersen & Graff (1965a, 1965b), used in this lower secondary school had not been influenced by the 'New Math' movement. They were written in a traditional way; i.e. definitions, theorems, proofs, examples and exercises, and with no trace of problem-solving, investigations, guided reinvention or open-ended tasks. The philosophy of the text-books can best be described as mechanistic (Treffers, 1987), and since these text-books were extensively used, this philosophy was reflected in the teaching of the subject.

Teaching at a college of education

In the first half of 1972 this researcher was appointed as a mathematics teacher at a college of education, and he started to teach there in August 1972. The teaching style during the two, three first years at the college was different compared with the teaching style used at the lower secondary school. At the college lecturing was the predominant teaching style. In that period the teaching philosophy shifted to the structuralist view as defined by Treffers (1987).

The National Curriculum for Compulsory Education in Norway, M74, was sanctioned in 1974. The mathematics curriculum in this national curriculum contained a section which focused on the working processes in the subject. This section emphasised the importance of inductive work in school mathematics, but at the same time the curriculum stated that "teaching mathematics where the deductive element is missing, is unthinkable" ((The) Ministry of Education and Research, 1974, p. 145).

A mathematics teacher at a college of education was responsible for introducing *M74* to the students. That implied that it was necessary to focus on why, and how to work inductively in mathematics. This had in turn impact at the personal philosophical view of the nature of mathematics, which gradually changed, and the empiricist view, as described by Treffers (1987), became also part of a personal philosophy.

Another milestone in the development of this personal history was a course, hold in August 1977, for mathematics teachers at tertiary education, of whom most of the participants were teachers working at colleges of education. One activity at this course was collaborative group work on what was called open-ended tasks or investigations. One of the tasks was the diagonal in a rectangle problem: A rectangle has dimensions $m \times n$, where m and n are natural numbers. How many of the mn square units in this rectangle will the diagonal pass through? For many of the participants at the course, this researcher included, this was a new type of tasks, and also a new type of mathematical activity. Still it is remembered how the group, in a way, 'stumbled along' when they started out working on this particular task.

At the same course educational material developed at the IOWO institute in the Netherlands was presented.[1] This material contained tasks or problems, which revealed a new dimension of mathematics and mathematics education. Personally, the investigational tasks, the collaboration, and the material was a new and positive experience, and in retrospect it is realised that this experience gradually influenced the personal view of the nature of mathematics, and hence also the teaching of the subject.

Looking at the problems constructed to the written examinations at the college, it is observed that investigational tasks appeared towards the end of the 1970's. In the period October 1978 - August 1990, when the competition started, nearly all of the written examinations, contained at least one investigational task. These tasks were more or less tailored to the same pattern; guided explorations followed by conjecturing and lastly and most often the students were encouraged to prove the conjecture.

The 1980's can be characterised as a problem-solving decade for school mathematics. There was a world-wide recognition that this subject had to be in the focus for school mathematics (see for example Burkhardt, Groves, Schoenfeld & Stacey (1984)), and Norway was in this respect no exception. From the beginning of that decade, problem-solving was on the agenda at the college. This was some year in front of the changes in the curriculum for compulsory education in Norway. A temporary (draft) version of the curriculum for compulsory education in Norway was published in 1985 (Grunnskolerådet, 1985), and the decisive curriculum, *M87* ((The) Ministry of Education and Research, 1987, 1990) was sanctioned in 1987.[2] In many respects the mathematical curriculum in *M87* differs from the mathematical curriculum in *M74*. One important reason for this difference was the rapid development of new technology which also could be used in mathematics. That will not be discussed in this work. Another important reason for the discrepancy was the impact of new knowledge due to increased research in mathematics education. The main topics in the *M74* mathematics curriculum were all pure mathematical topics such as: Numbers, functions, algebra, geometry etc.. In the mathematics curriculum of *M87*, on the other hand, the first main topic was problem-solving. Problem-solving is not a pure mathematical topic, it is a topic which is related to the working process in the subject. In addition *M87* stressed and emphasised that the other topics in this mathematics curriculum should be taught according to a problem-solving perspective. Compared with the perspective in *M74*, this was a new perspective.

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[1] IOWO [Institute for Development of Mathematics Education] "was the national institute for the development of mathematics education in the Netherlands from 1971 through 1980.... In 1980 the IOWO was terminated, but the research activities were continued in Research Group on Mathematics Education and Educational Computer Centre (OW&OC), that was renamed Freudenthal Institute in 1992 to honour its founder Hans Freudenthal" (Gravemeijer, 1994, p. 12 and p. 15).

[2] The national curriculum for compulsory education in Norway was sanctioned by Stortinget (the Norwegian National Assembly) in the spring 1987. The Norwegian edition of the curriculum was published in August 1987. An English version was published in 1990. It is common to refer this national curriculum for compulsory education in Norway, as *M87*.

Until August 1992 mathematics was a voluntary subject in teacher education in Norway. Many of the students at the college of education had what may be characterised as ‘bad’ experiences, from their previous mathematics education (primary or/and secondary school). Most of them had low marks in mathematics, their success in the subject had been what may be called limited, they felt ‘insecure’ working with this subject, and as a consequence of this their attitude and feelings to mathematics were negative. Even though mathematics was a voluntary subject, several of those students, nevertheless, selected mathematics as one of the subjects to be studied at the college. One reason for this choice was that a complete study from a Norwegian college of education qualified the student to teach all subjects in all grades in the Norwegian compulsory school.

The students studying mathematics had usually experienced mathematics as a subject consisting of facts and rules which had to be used in accordance to some more or less unintelligible methods; i.e. a mechanistic approach. Personally, it was not believed that the Norwegian school or a large part of the mathematics teachers in that school to be so unprofessional that they ‘produced’ college students with very little or limited knowledge in mathematics. When the students entered the college they had been at school for at least 12 years, and for at least 10 of those years they had mathematics as one of their main subjects, they must certainly have ‘learned’ some mathematics. The personal opinion was that an average or normally equipped college student should not have serious problems with the mathematics at a college of education. Nissen (1994) maintains the same point of view concerning the mathematics in secondary schools in Denmark. The personal opinion was that the picture as described above, did not tell the whole truth concerning the college students’ mathematical knowledge. A large group of them certainly had difficulties with the mathematical language, and in handling symbols. Their formal knowledge in mathematics had large gaps, or was low. However, the personal process undergone since 1972 had gradually changed the view to this researcher on the nature of mathematics, and the conviction was that mathematics was more than just handling symbols, it was also a search for patterns and relationships.

Since the middle of the 1970’s the mathematics students at the college had now and then got the opportunity to work inductively, and also to collaborate on solving problems. After the 1977 summer course, the use of investigational tasks became part of the teaching practice at the college, and the students were also encouraged to collaborate when working on mathematical problems. It was observed that most of the students which had problems using the mathematical language, and in handling mathematical symbols, were able to handle these investigational tasks in a meaningful way. They discovered and described patterns and relationships, but mostly in an informal fashion. These students had acquired informal knowledge in mathematics, but had a gap to the formal mathematical language. The personal experience was that a student with little formal knowledge in mathematics usually had a negative attitude to mathematics. Facing those students it was realised that the first challenge was to ‘work’ with their attitudes and feelings to mathematics, and then, secondly, try to bridge the gap between their informal knowledge and the formal mathematics. The challenges were: How should the teaching be organised if the students should change their attitudes and feelings about mathematics from negative to positive, and how to prepare the ground so it could be possible for the students to build ‘bridges’ from their informal knowledge to the formal mathematics?

After some years of teaching practice the personal conviction was that a teaching strategy which focused on symbols and their manipulations, was not the right one. This strategy had been used, and in most cases it had failed. For a large group of students it seemed that explanations or expositions had very little effect on their learning and understanding of mathematics. It was necessary to find and use other teaching strategies. Jaworski (1994) asserts the same experience.

As mentioned above, the conviction was that the students had informal mathematical knowledge. If the students themselves could realise that they had mathematical knowledge, would that have an impact on their attitude to mathematics? During the teaching practice it was observed that investigational activities, for most of the students were challenging and engaging. That was a possible starting point, and the following strategy was adopted:The mathematics students at the college should start their mathematical education with investigational tasks which they had to work collaboratively. The form and the content of these tasks had to be different from the tasks or exercises the students had experienced in their previous mathematical education.

One consequence of this strategy was the necessity to find or construct tasks where the entrance level ‘into’ the task was so low that the students were able to work meaningful with the task. This, in turn, implied that the formal mathematics required to carry through the tasks could not to be to complicated or to abstract for the students, which meant that it was necessary to be careful with the language and the symbols used. In addition, the mathematical activity of the tasks had to be meaningful (and of importance), and should give opportunity to make generalisations. These generalisations could possibly be a bridge to formal mathematical knowledge. A brief characterisation of such tasks is that they had to give opportunity to solutions on different levels, and they should have an open ending. The outcome of adapting this teaching strategy was, as a whole, more successful than a teaching strategy which focused on symbols and their manipulation. The students were in most of the cases able to discover and explain patterns and relationships using their own words, but frequently they were unable to express the same relationships using mathematical language and symbols. However, with some assistance, it was possible for most of the students to build some bridges into formal mathematics. Another important result was that nearly all the students had changed their attitude to mathematics to a more positive one. Some bridges had been build, but that did not mean that the students had been ‘mathematicians’.

The preceding is a short description of this researchers pedagogical development and his pedagogical experiences until the beginning of 1990 when the first germ to this thesis was sown, however at that time a germ that was invisible for this researcher.

1. BACKGROUND AND RESEARCH QUESTIONS

1.1 Introduction

Research perspectives shifts. During the 60's and 70's much of the research in mathematics education focused on the individual child learning mathematical concepts where the setting had been the child in an isolated situation; i.e. the context was not the classroom it was the 'laboratory'. Then gradually during the 1980's more and more of the research focused on children's learning of mathematics or mathematical thinking where the context was the classroom. The shift in research perspective was from the individual child in an isolated situation to the children in a social situation. It was in that period, 1990, the mathematics didactics milieu in Bergen, where Professor Stieg Mellin-Olsen was in the forefront, founded a journal, *The TANGENT. Journal for mathematics in compulsory school* [*TANGENTEN*, *Tidsskrift for matematikk i grunnskolen*]. In texts the version *Tangenten* is the most frequently used, and it will be used in this study. The subtitle, *Journal for mathematics in compulsory school*, indicated that *Tangenten* was meant for teachers teaching grades 1-9, that is teachers in primary and lower secondary school. As a consequence of the 1997 school reform in Norway, the length of the compulsory education was increased from 9 to 10 years. The numbering of the grades used in this study is the old one; i.e. the numbering used until 1997.

The first issue of the journal was published in September 1990, with Mellin-Olsen as the first editor. In volume **5** number 2, **5**(2), the subtitle was changed to *Journal for mathematics* [*Tidsskrift for matematikk*]. This subtitle was in issue **6**(1) changed to *Journal for mathematics teaching* [*Tidsskrift for matematikkundervisning*], and in 2005, volume **16**, this is still the subtitle.

The founders of the journal decided that each number of *Tangenten* should contain a section which should be a competition for school-classes. This researcher was asked to edit this section, which meant responsibility for finding suitable tasks for the competition, for evaluating the pupils' answers, and for commenting on their work in the journal (Torkildsen, 1991a, 1991b, 1991c, 1991d, 1992a, 1992b, 1992c, 1993a). The column was titled *Ole Einar's task page. The Tangent's competition for school-classes* [*OLE EINARS OPPGAVESIDE. TANGENTENS KONKURRANSE FOR SKOLEKLASSER*].

The number of competing classes was never high, and due to this fact, the competition came to an end in September 1992. The last task, *Length of Trains*, appeared in *Tangenten* **3**(3). It was only the competition that came to an end, the publishing of *Tangenten* continued, and still, in 2005, four issues are published each year.

Mellin-Olsen gave some guidelines or demands the competition tasks should meet. A brief characterisation of the tasks is that they should be *open-ended*. This will be further discussed in chapter 3.

1.2 Mathematising

The evaluation of the pupils' answers revealed that some of the solutions were quite unexpected and surprising. This unexpectedness was manifested in different ways. Some of the works showed an originality and inventiveness in the way the pupils attacked the tasks, and hence their solutions, others showed originality in the way they argued. It was observed that the use of mathematical symbols, to a large extent, varied from solution to solution. There were instances where few mathematical symbols were used and others where several symbols were used. For a few of the solutions it appeared not to be a correspondence between the use of mathematical symbols and the mathematical richness of the solution. The use of few mathematical symbols did not necessarily imply that a solution could be classified as mathematical 'meagre' or 'poor'. Disregarding whether the pupils in their solutions used few or many mathematical symbols, the solutions were the result of an organising and structuring activity, *mathematising*. Mathematising is here used in the way defined by Treffers (1987, p. 247) "as the organising and structuring activity in which acquired knowledge and abilities are called upon in order to discover unknown regularities, connections, structures.".

According to Treffers (1987) mathematising has both a *vertical* and a *horizontal* component. The horizontal component relates to "transforming a problem field into a mathematical problem" (Treffers, 1987, p. 247), or in Freudenthal's words "leads from the world of *life* to the world of *symbols*" (Freudenthal, 1991, p. 41). The vertical component relates to "processing within the mathematical system" (Treffers, 1987, p. 247), or to stay in 'the world of symbols' where "symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly" (Freudenthal, 1991, p.41-42). However, Freudenthal (1991) stressed also that there is no clear distinction between these two directions, the distinction is dependent on several factors as the specific situation, the person involved, and the environment. The mathematical richness in the pupils' work could be classified according to those two directions, the vertical and/or the horizontal. The horizontal component, which transforms a problem into a mathematical problem, lay the foundation for using a mathematical language, and hence the use of mathematical symbols. The vertical component elaborates and stays inside the 'world of symbols'. This means that the horizontal component ends with or inside 'the world of mathematical symbols' and the vertical component is inside that world.

Related to the context the tasks can be sorted into two categories. Some tasks had their context inside the 'world of mathematics', others draw the context outside that world. In the *Tangenten* articles that commented on the pupils' solutions (Torkildsen, 1991a, 1991b, 1991c, 1991d, 1992a, 1992b, 1992c, 1993a) these two directions, the horizontal and the vertical were, to a greater or lesser degree, present. Which component, and the degree of presence of these two components, depended mainly on the task. For the tasks of the inside category only the vertical component was focused in the articles. Both components, the horizontal and the vertical, were present in articles that commented on tasks from the outside category.

The structure in the *Tangenten* articles was more or less the same. For the actual task at least one formal solution was given, for some tasks several solutions were presented. The number of solutions given was dependent on both the task and the number of different solutions given by the classes. In these articles the pupils' solutions were compared or related to the(se) formal solution(s). In order to simplify this comparison, sections of pupils' solutions were translated or decoded into a formal mathematical symbolic language. The result of this decoding process was a product in which the mathematics or the mathematical richness used by the pupils in their solution procedures became more visible than in the original work. The mathematical symbolic language un-earthed or brought to light, to a certain extent the mathematics used by the pupils in their solution procedures. This decoding process could also be characterised as *mathematical archaeology*.

1.3 Mathematical archaeology

The notion mathematical archaeology is a young one. As far as known it was first introduced by Ole Skovsmose in 1994 in his book, *Towards a Philosophy of Critical Mathematics Education*. He asserts that this notion can be given two different interpretations, a *global* one, and an *educational* one. By the global interpretation he understood excavating or bringing to light the mathematics hidden and used in "social structures and routines" (p. 95). In a later paper Skovsmose described mathematical archaeology as "… the process of excavating mathematics which might be encapsulated in certain political arguments, technologies or administrative routines." (Skovsmose, 1999a, p. 14). A process of clarifying or make visible the mathematics that in many instances in our society has become a nearly tacit and invisible actor in decision-making processes, is an example of mathematical archaeology. The global interpretation of mathematical archaeology can be characterised as reflecting on the socio-political dimensions of mathematical models being developed or applied.

The educational interpretation rests on the fact that even in educational settings mathematics can be hidden or invisible to both pupils and/or teachers. Mathematics can be so integrated in the pupils' activities that it was not realised, neither by the pupils nor by their teacher that the pupils were doing mathematics when the activity was performed, or after the performance of the activity they were not able to identify the mathematics carried out. Skovsmose gives some examples of project works where this has happened. In order to be able to identify the mathematics in such contexts, Skovsmose (1994a) points out that it therefore will make sense to carry through a mathematical archaeology in an educational setting. Mathematical archaeology in the classroom should focus on the fact "…that some of the activities carried out in the classroom – (…) – are in fact mathematics." (Skovsmose, 1994a, p. 96).

1.3.1 Socio-political dimensions

The importance for why "Mathematics has to be recognised and named" (Skovsmose, 1994a, p. 94), is linked to the fact that mathematics is a vital part of our daily life, and as such has what he calls *a formatting power*, i.e. "the idea that mathematics can influence, generate or limit social actions" (Skovsmose & Yasukawa, 2000, p. 2). The aim of a mathematical archaeology with a socio-political focus is to un-earth or clarify possible socio-political consequences of the mathematical models questioned, and thus establish a basis for intervening or affecting on the decision-making processes based on actual mathematical models.

The impact, and to many people invisible impact, of mathematics on society is stressed by several mathematics educators (e.g. Blomhøj, 2000; Jørgensen, 1998; Niss, 1990, 1994; Skovsmose, 1999b; Skovsmose & Yasukawa, 2000; Tymoczko, 1994). In an article Niss (1994) stressed that mathematics, on a variety of arenas, plays several important roles. The discipline is applied in many other disciplines (e.g. physics, economies, information science, etc.), it is involved in several sectors of social practices (e.g. decision-making, prediction, design, etc.), and it is part of everyday life (e.g. transactions, graphical representations, codes, etc.). One of his conclusions is "… mathematics contributes in a thorough way to the *shaping of society*," (Niss, 1994, p. 370). If the roles and functions of mathematics in our societies are hidden or invisible to a large group of the citizens, it will not be possible, or at best it, will be very difficult for this group of people to participate actively in decision-making processes. Applications based on decision-making processes, which in turn is based on mathematical models, will be out of control for those citizens. One challenge for the mathematics educators is therefore to make visible or un-earth (some of) the mathematics used or applied by society. This has in turn an impact on the mathematics curriculum, and during the last two decades the importance of mathematical models has also found its way into mathematics education. This has been and is still reflected through the important place given to this subject in the mathematics curriculum in a vast majority of countries (Niss, 1996). See for example the present mathematics curriculum for upper secondary education in Norway ((The) Ministry of Education, Research and Church Affairs, 2000), and the proposition presenting the next mathematics curriculum for the upper secondary school published in June 2005 (Utdanningsdirektoratet, 2005).

One important challenge for mathematics educators is the justification for the mathematics curriculum in the schools. A mathematical archaeology that excavates the position and the role mathematics is playing in society is an important tool for justifying the mathematics curriculum, and a necessity that these curricula are at a satisfactory level. The justification debate is found to be outside the scope of this work, and will therefore not be discussed further in this context. Discussion about the justification of the mathematics curricula is found in e.g. Ernest (1991, 1998), Jørgensen (1998), Niss (1990, 1994, 1996), or Winsløw (1998).

Niss in the 1990, 1994 or the 1996 articles strongly emphasised the importance of making the mathematics used in extra-mathematical fields visible; i.e. un-earth this mathematics. The arguments used, could be characterised as advocating a mathematical archaeology where the socio-political dimension plays an important role. This focus was also advocated by the Danish initiative, *Mathematics Teaching and Democracy* (Nissen, 1990). A mathematical archaeology reflecting on the socio-political dimensions of mathematical models is an important dimension, but it is found to be a narrow focus. One argument for maintaining this is the restriction of doing the mathematical archaeology on mathematical models, whether such a model is developed and applied by the society in the decision-making processes or has been developed in an educational setting. In this argument is also included the case of focusing on the socio-political dimensions on educational activities not originally designed and carried through in a mathematical context or on a mathematical background, for example, project works.

1.3.2 Educational dimensions

Skovsmose (1994a, 1999a) argues that the focus of a mathematical archaeology in a non-mathematical context can also be 'pure' mathematical. In the classroom a mathematical archaeology can focus on the pure mathematics inherent in pupils' activities or the result of such activities. As an example Skovsmose (1994a, p. 95) mention that the children in the project work 'Constructions' realised that in the construction of a bridge, cubes were unstable elements in contrast to the stable triangles. The pupils, however, did not follow up this discovery, which could have been used as a basis for further investigations in pure mathematics, in this case in geometry. In other contexts a mathematical archaeology could lead to different mathematical arenas.

Outside the classroom ethnomathematical research programs in many instances exemplifies a mathematical archaeology focusing on the pure mathematics inherent in activities carried out in a non-mathematical context; e.g. street mathematics (Nunes, Schliemann & Carraher, 1993), mathematics found in decorations (Gerdes, 1995b), mathematics found in handicrafts as the weaving of baskets (Gerdes, 1986), knitting (Mellin-Olsen, 1993a), etc. A more comprehensive overview of ethnomathematical studies is found in Gerdes (1996).

A mathematical archaeology carried out in a non-mathematical context or setting is important regardless of whether the focus is socio-political or pure mathematical. However, an archaeological excavation can also be made in a *mathematical* context. An example from pure mathematics is Abel's proof of the continuity theorem. In this proof "Several mathematicians … have seen the germ of the concept uniform convergence" (Bekken, 2003, p. 19). The concept was however defined several years after Abel published his proof.

A mathematical archaeology in a mathematical and educational context can be done on at least two different arenas. It can be used in order to uncover the pure mathematics that is hidden or embedded in *pupils*' mathematical works, but it can also be used for analysing teaching elements constructed by *educators*. The first one of these two arenas, the pupils' works, is heavily linked to practice. The archaeology is carried out on the result or at the outcome of the implementation of a teaching element. This study has its focus in this arena. The archaeology in the second arena is carried out directly on the teaching element, and it is therefore not based on observations or results from practice. However, the outcome of the excavation can have implications for practice. This arena will not be the arena for this study.

Mathematical archaeology on pupils' mathematical works

The overall question in mathematics education is how to bring the area forward. Wittmann (1995) argues that this cannot be done without using the close relationship to disciplines as mathematics, psychology, pedagogy, and sociology. All these areas are well-established disciplines with their standards and methods, and just to copy or adopt some of these standards or methods in mathematics education may represent a risk. The reason for this is the risk for undermining the applied nature of mathematics education. In order to avoid or reduce this risk it is necessary that the researchers have proximity with practice. As put by Wittmann:

> …the specific tasks of mathematics education can only be actualized if research and development have specific linkages with practices at their core and if the improvement of practice is merged with the progress of the field as a whole. (Wittmann, 1995, p. 356)

A basic principle in teaching mathematics, or one could also call it an axiom, is to base the teaching of the subject on the knowledge to the learners. Consequently, it is of vital interest, to the greatest possible extent, to un-cover this mathematical knowledge. Quite naturally a large part of the research in mathematics education has therefore been grappling with the learners' level of knowledge. This research has used a variety of different methods; i.e. tests, inquiries, classroom observations, interviews, text analysis etc. In his article Wittmann (1995) specified eight components of this core that are of particular interest. The first component on this list is "analysis of mathematical activity and of mathematical ways of thinking" (p. 356). Analysing the pupils' written mathematical work is one of the possible sources for the *explicit-making* of the mathematics inherent in these works. Such an analysis can be one source that can give valuable information or knowledge about the pupils' or students' mathematical activity and their mathematical thinking.

Mathematical explicit-making

The research in mathematics education covers a broad spectrum of subject areas. One of these is the pupils' or students' achievements in the subject. The research foci in this area can be looked upon from different perspectives, two of these perspectives can respectively be characterised as success, and mistake. Under the first perspective, success, the point of view for the research is 'what they can do'. This perspective focus on what the pupils have done, what they were able to do. It is the strength of the achievement(s) that is highlighted. This is a perspective that points at the possibilities, what is possible to achieve, and it is the perspective for this particular research. This is an approach which is advocated by Hughes (1986). Concerning the second perspective, the aspect of the research can be described as 'what the pupils can't do'. The focus is in this case often directed towards what the pupils were not able to do, contrary to what was supposed and expected. The outcome of that type of research has been a strong argument for the more or less repeated and recurring 'back to basic' movements in the US (O'Brien, 1999). It is a fact that, for a substantial part of the previous century, research projects was directed towards the uncovering of errors, often computational errors, or misuse of computational procedures (Jones, Langrall, Thornton & Nisbet, 2002). As expressed by Freudenthal (1993) "…hunting for errors has, most of the time, been a favourite sport in research on mathematics education" (p. 72), however, he continued "but its reputation, though still high, has been declining for a few years." Since then, 1993, this 'hunting of errors' has further declined. One reason for this declining is that computational skills and performing computational procedures no longer is given status as 'powerful mathematics' (Jones, Langrall, Thornton & Nisbet, 2002). However, having in mind the discussions in media when test results are published, does not confirm that the 'hunting of errors' has lost its reputation. Two recent examples from late 2004 is the debate in Norway that followed after the presentation of the results of the *Trends in International Mathematics and Science Study*, TIMSS, and the *Programme for International Student Assessment*, PISA.

As mentioned in paragraph 1.2; the extent to which the classes applied mathematical symbols in their solutions varied, and in some instances it appeared that there was not correspondence between the use of mathematical symbols and the mathematical richness of the solution. In order to throw light on the mathematics present in the solutions, it is necessary to make this mathematics visible; to look at what the pupils actually have done in their mathematical texts. The question is then if mathematical archaeology can be one way or method to excavate and give explicit mathematical descriptions of the hidden or underlying mathematics found in the pupils' mathematical texts.

1.4 Theoretical framework

It should be noticed that when the pupils' solutions were commented in *Tangenten* (Torkildsen, 1991a, 1991b, 1991c, 1991d, 1992a, 1992b, 1992c, 1993a), it was not the intention that nearly all of these solutions later on would be used as a basis for an academic work, a doctoral thesis. At the time when the commenting articles were written, it was therefore not a question of an explicit reference to a theoretical framework. This question however, became important when it was decided to do an academic work. The situation or the background was then:

- the data were collected; the pupils' solutions;
- the goal or the objective was there; un-earth the mathematics in these solutions.

The question was then the theoretical framework. Related to the described background this question can be formulated as:

> *How can the mathematics un-earthed from the pupils' solution procedures be characterised*?

The considerations from section 1.2 indicate that the theoretical framework could be found in mathematics. The body of mathematics, however, is huge; it consists of a multitude of topics, topics that more or less are interwoven in each other. The question of framework was then transformed to focus on what topic or part of mathematics could be suitable or adapted to this case. Further study of the results of the decoding process revealed that *mathematical structures* could be a suitable theoretical framework.

Mathematical structures have played and play an important role in mathematics, and hence also in mathematics education. Mastering and understanding of mathematical structures is an important goal for mathematics education. Teaching of mathematics is in a way to move from simple mathematical structures to complex mathematical structures.

1.5 Research question

On the background of the considerations from sections 1.1-1.4 the following two research questions are formulated:

1. ***What kind of mathematical structures are inherent in the solution procedures that pupils have applied in some of the open-ended tasks used in the Tangenten competition?***

and

2. ***Is mathematical archaeology applied on pupils' written solutions of open-ended tasks a suitable tool for increasing the knowledge about pupils' mathematical activity and their mathematical thinking?***

1.6 Delimitation

Mathematical structures can be classified as more or less *complex*. Mathematical structures like natural numbers and order, natural numbers and the four basic arithmetic operations etc. could be classified as less complex. Children's or pupils' relations to these mathematical structures have been discussed in many papers and they will not be dealt with in this connection. See for example (Brekke, 1991; Carpenter, Franke, Jacobs, Fennema & Empson, 1997; Dockrell & McShane, 1992; NCTM, 1978; Pepper & Hunting, 1997; Vergnaud, 1983,1988; Vergnaud, Rouchier, Ricco, Marthe, Metregiste & Giacobbe, 1981; Verschaffel & De Corte, 1996).

1.7 Structure of the thesis

The thesis is divided into 11 chapters and an appendix A which is numbered chapter 12.. Chapter 1 focuses on the background, the context of the study and the research questions.

Mathematical structures which are the theoretical framework are the theme for chapter 2. The chapter is organised in three main parts. The first part presents a brief historical development of mathematical structures. The second part deals with the use and understanding of mathematical structures in mathematics education. A particular weight is attached to Freudenthal's works. The third part gives a definition of the term and some examples of mathematical structures are given.

Chapter 3 focuses on the pedagogical philosophy on which the competition was based, and on the constructed tasks. In the first half of the chapter this philosophy is mirrored through the competition rules which have a particular weight on collaboration and communication. The second half concentrates on the concept open-ended tasks or investigational tasks.

Chapter 4 explains the research methodology. This includes the data collection and analysis method and the discussions on reliability and validity.

Analysis of the pupils' solution procedures is the focus for the chapters 5 – 10. There is one chapter for each of the tasks included in this study. The analysis chapters have the same structure. The first section demonstrates formal solution(s) of the tasks. This section can be considered as a continuation of the articles in *Tangenten* (Torkildsen, 1991a, 1991b, 1991c, 1991d, 1992a, 1992b, 1992c, 1993). For most of the tasks there are several possible alternatives for solution procedures. The intention of this part was not to demonstrate as many as possible of these alternatives, but to mirror the different solution procedures applied by the participating classes. The remaining sections of the analysis chapters concentrate on the analysis of the pupils' solution procedures.

The findings of the analysis, the answer to the research question, are reviewed and discussed in chapter 11.

Appendix A gives an overview of the competition rules and the changes and revisions of the rules that took place during the competition period.

Copies of the classes' solutions are available on the enclosed CD.

Two concepts are central in the research questions; mathematical structures and open-ended tasks. In the next chapter the first of these two concepts, mathematical structures, is addressed.

2. MATHEMATICAL STRUCTURES

Mathematical structures are the theoretical framework for this study. The chapter is organised in three main parts; the historical development of mathematical structures, mathematical structures related to mathematics education and the definition of mathematical structures in this study. In the education part special weight is attached to Freudenthal's works.

2.1 Introduction

The origin of the word *structure* [*struktur* in Norwegian] is the Latin word *structura*, which means building or joining (Store Norske, 1989). It is a frequently used word in both the English and the Norwegian languages. *Webster's Enclyclopedic Unabridged Dictionary of the English Language* gives several explanations or definitions of this word. The main one, ranked as number one, is "mode of building, construction, or organisation; arrangement of parts, elements, or constituents" (Webster's, 1989, p. 1410). In addition several other spheres is mentioned as examples where the word structure is used; e.g. biology, geology and sociology. Mathematics is not mentioned as one of the spheres. This in contrast with statements as:

- "one of the fundamental concepts of modern mathematics" (Collier's, 1985, p.547);
- "a great many - some would say all - mathematical statements are about structures" (Bell & Machover, 1977, p. 9);
- "Mathematics can be defined as the theory about structures on sets" (Thompson, 1991, p.396);
- "Mathematics consists of structures, and their associated models and symbol systems"(Bell, 1976, p. 2.6).

2.2 Mathematics

2.2.1 History

In the history of mathematics, the notion of structure is a young one. It belongs to the mathematics of the twentieth century, nevertheless it is asserted that "the origins of structuralism can be found in many theories of the 19th century, but its development is a characteristic of the second part of our century" (Speranza, 1994, p.171). The mathematics of the 20th century is characterised by its structural character (Corry, 1992, 1996). The notion of structure or related notions, e.g. system, have a central position in modern mathematics (Aubert, 1973; Bomann, 1979; Collier's, 1985; Stone, 1961).

Among mathematicians it is agreed upon that algebra was the first mathematical discipline where the structural approach appeared, and that the evolution of mathematical structures is closely linked to the evolution of modern algebra, which is characterised by what is called algebraic structures; e.g. groups, rings, fields, vector spaces, etc. (Aubert, 1973). According to Corry (1992, 1996) the book, who first presented algebra from a structural point of view, was *Moderne Algebra* (1930) by B. L. van der Waerden. In the introduction to his book, van der Waerden pointed out that the building of concepts (*Begriffsbildung*) in algebra has been numerous, and this in turn has given knowledge of new relationships and has lead to far reaching results. The objective of this book was to guide the reader into this new world of concepts, or as putted by Corry "*…to define the diverse algebraic domains and to attempt to elucidate fully their structure*." (1996, p. 46). Van der Waerden did not use the term mathematical structure in his book, but still the book is about mathematical structures. For more details see Corry (1996).

An attempt to develop a formal theory of structures was done in a research program initiated by Øystein Ore in the beginning of 1935. The objective of the program was "to develop a general foundation for all abstract algebra based on the notion of lattice, a notion which he (Ore) denoted as **structure**." (Corry, 1996, p. 263). The programme did not capture enough interest among the researchers and after about 10 years it was abandoned (Corry, 1996). In the same period a new actor entered the stage, *Nicolas Bourbaki*.

2.2.2 Bourbaki

Nicolas Bourbaki is a pseudonym for a group of mathematicians in France. This group was organised in the middle of the 1930's and consisted at that time, of young and mostly French mathematicians.[3]

The literature asserts two main reasons for the establishing of the Bourbaki group. A group of young French mathematicians was, after the World War I not satisfied with the mathematics teachers and the traditional mathematics textbooks used in the country. They wanted therefore to create new textbooks, and they also wanted to re-establish the former existing universality of French mathematics (Blomhøj, Frisdal & Olsen, 1984; Corry, 1992, 1996; Mandelbrot, 1994). To reach these goals they intended to write a comprehensive treatise, *Éléments de Mathématique*. In the preface to the volume *Theory of Sets*, Bourbaki (1968), it is usual to refer to the Bourbaki-group as one person, wrote that the main purpose of the treatise "is to provide a solid foundation for the whole body of modern mathematics." (p. v). Each volume has been written in accordance to this purpose, and the result is books written in a contemporary and original fashion which illustrate the axiomatic structure of

[3] B. Mandelbrot says in an interview that the group was "a militant bunch, with strong biases against geometry" (Barcellos, 1985, p.210).

modern mathematics (The New Encyclopædia Britannica, 1990). For more details about the Bourbaki group see e.g. Blomhøj, Frisdal & Olsen (1985); Borel, (1998), Corry (1992, 1996), Dieudonné (1970), Halmos (1957), Kahane (1998), Katz, (1993), Mandelbrot (1994), Skovsmose (1981), Walton (1990).

A natural question is then: What about the influence of Bourbaki on mathematics? Many of the books written by Bourbaki have been classic, and the influence on the "mathematical activity (research, teaching, publishing, resources distribution) has been enormously significant." (Corry, 1996, p. 298). However, Corry (1992, 1996) maintains also that the influence on contemporary mathematics is "an arduous task". Blomhøj, Frisdahl & Olsen (1985) asserted that Bourbaki might have had some influence on mathematics. Freudenthal says it more strongly:

> Bourbaki's work is a monumental system of mathematics, which -- although now out of date on essential points -- has contributed enormously to the growth of mathematics. (Freudenthal, 1991, p. 24)

This is supported Kahane (1998), who maintain that "Bourbaki is out of fashion. However, ...the work of Bourbaki remains a masterpiece of our time." (p. 81). Schwartz (2001) compare Bourbaki's work with the revolution in biology caused by Linnæus's work *Systema naturae* from 1758. The overall picture among a large group of mathematicians is that Bourbaki has had a great impact on the development of mathematics (Collier's, 1985; The New Encyclopædia Britannica, 1990; Store Norske, 1989; Walton, 1990). Corry (1992, 1996) gives a more differentiated picture, the degree of influence varies with the mathematical disciplines, period of time and country. Topology and algebra are probably the most influenced disciplines, while logic and most of the applied mathematics are not influenced at all.

2.2.3 Bourbaki and mathematical structures

From a historical point of view the notion *mathematical structure* is very often associated or identified with Bourbaki. The idea of Bourbaki was to present the whole of mathematics in a unified comprehensive and multivolume treatise. In this context the concept of structure was intended to play an important or conclusive role. This is emphasised by Bourbaki (1950), Dieudonné (1970), and Kahane (1998).

In the book *Theory of Sets* (1968), Bourbaki gave a precise definition of the term *structures* or the term *species of structures*, as is used in the book. The intention (purpose) of this term is explained in the introduction to this volume, the objective is to define a concept that will be suitable to classify complex mathematical objects, or as put by Bourbaki "to classify them according to the *structures* to which they belong." (p. 9). First at page 262 in this volume the definition of structures is presented, and without reading most of the text before this definition, it is not easy to grasp the meaning. Blomhøj, Frisdahl & Olsen (1985) have pointed at a simplification of this definition is found in Bell & Machover (1977).[4]

Bourbaki identifies three basic types of structures, the so-called *mother-structures*, which are:

- order structures;
- algebraic structures;
- topological structures.

Each of the three mother-structures is or can be organised in a hierarchy of structures, from poorer to richer. A structure becomes *richer* when for example the number of axioms within a structure is increased (Blomhøj, Frisdahl & Olsen, 1985; Corry, 1992, 1996). A *multiple structure* is constructed if two mother-structures are "combined organically by one or more axioms which set up a connection between them" (Bourbaki, 1950, p. 229). This is what Wittmann (1978) called 'Spezialiserung' and 'Überlagerung' (Mischung).

The picture of mathematics as a hierarchy of structures "belong strictly to Bourbaki's images of mathematics" (Corry, 1992, p. 340). This is also mentioned by Ernest (1991) and Niss (1996).

Among mathematics educators there are highly agreement that Bourbaki gave inspiration and incitement to the *New Math* movement that took place in the period 1955 - 1970. This is noticed by many authors (e.g. Blomhøj, Frisdal & Olsen, 1985; de Lange, 1987, 1996; Hanna & Jahnke, 1996; Kahane, 1998; Katz, 1993; Niss, 1996; Skovsmose, 1981), and it is emphasised by Lerman:

> The New Maths changed little, except to make the structure of the mathematics curriculum parallel to the logical structure of mathematics as constructed by Bourbaki. (Lerman, 1994, p. 3)

2.2.4 Critics of Bourbaki's structure concept

In the article 'Nicolas Bourbaki and the Concept of Mathematical Structure' Corry (1992) maintains (p. 315) that it was "a superfluous undertaking" by Bourbaki to introduce structures. He argues that Bourbaki in his publications has used the term structure both in a non-formal and a formal setting, and that Bourbaki in his proofs did not make use of his definition of mathematical structures, even if he used the term. His conclusion is that the concept of structures is not essential to the Bourbaki writings, especially the *Éléments de*

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[4] The definition is found at page 9 and it is formulated:

A structure consists of the following ingredients:

(1) A non-empty class, called the universe *or* domain *of the structure. The members of this universe is called the* individuals *of the structure.*

(2) Various operations on the universe. These are called the basic operations *of the structure.*

(3) Various relations on the universe. These are called the basic relations *of the universe.*

Mathématique. This critique is maintained in Corry (1996), and it is emphasised that the idea did never become the central pillar it was singled out to be in Bourbaki's mathematical universe. More than that, Corry maintains that this concept is used and understood in many ways, "it is interpreted differently by different mathematicians."(Corry, 1992, p. 317). His observation is that "this idea [of structure] belongs to the corpus of tacit knowledge shared by mathematicians" Corry (1992, p. 317), which means that the idea is non-formal. He then concludes that in contemporary mathematics the term mathematical structure is only used in a non-formal way.

2.3 Mathematical structures and education

Mathematical structure or mathematical structures are terms that are frequently used also in mathematics education (the didactics of mathematics). In that sphere, these terms or notions are often connected or linked to cognitive structures; e.g. Dienes & Jeeves, 1965; Dreyfus, 1991; Schoenfeld, 1992; Skovsmose, 1981; Wittmann, 1978. For example Wittmann (1978) mention that Jean Dieudonné and Jean Piaget in 1952 for the first time pointed out the connections or links between the mother-structures to Bourbaki and the psychological theory to Piaget. Later on this view has met heavily critics (Corry, 1992, 1996; Freudenthal, 1973, 1991; Skovsmose, 1981). This work will not be concerned with the discussion or critique of the link between a formal mathematical theory, as manifested by Bourbaki, and the theory of Piaget. What are of interest for this work is what mathematics educators mean or understand with the term mathematical structure or mathematical structures.

Mathematical structures had a very strong position in the New Math reform programs. After the 'fall' of New Math in the 1970's it seems that the notion mathematical structures for a period more or less was avoided by the researchers in mathematics education. Gradually, from the beginning of the 1980's, the term mathematical structures was again used by the researchers in mathematics education, it is used by different authors and in a broad spectrum of areas in mathematics education. A mathematics educator that actively used this term in the 1980's, was *Hans Freudenthal*.

2.3.1 Freudenthal

Hans Freudenthal (1905 - 1990) started his career in pure and applied mathematics. In 1955 he became a member of *The International Commission on Mathematics Instruction* (ICMI), from that time "Freudenthal's change of perspective from mathematics to mathematics education became obvious" (Streefland, 1990, p. 600). Freudenthal wrote four major works on mathematical education:

- *Mathematics as an Educational Task*;(1973)
- *Weeding an Sowing* (1978);

- *Didactical Phenomenology of Mathematical Structures* (1983);
- *Revisiting Mathematical Education. China Lectures* (1991).

Besides these four books is found “an endless stream of papers, lectures and articles” (Goffree, 1993, p. 30). The list of Freudenthal’s publications in mathematics education as presented in *Revisiting Mathematical Education. China Lectures,* contains 201 titles. In addition he founded and was the first editor of the journal *Educational Studies in Mathematics Education*. This means that Freudenthal for a period of about 35 years had a central and influential position in mathematics education.

Mathematical structure was one of the many subjects (topics) that Freudenthal engaged in. As far as can be seen, he was the first mathematics educator who raised and discussed on a broad basis, and in an educational setting, the meaning of the term mathematical structures. This discussion is found mainly in his last two major works, *Didactical Phenomenology of Mathematical Structures* (1983) and *Revisiting Mathematical Education. China Lectures* (1991). In these two books the focus is children and mathematical structures. It is therefore natural to take a closer look at Freudenthal’s works.

Freudenthal and Mathematical Structures

The structural character of mathematics was and is for many people, mathematicians as well as for non-mathematicians, almost a dogma. The New Math movement supported this view, even though the main goal for this movement was understanding. Understanding “was emphasised as the predominant goal of mathematics teaching at all levels, from kindergarten to university” (Niss, 1996, p. 31). To obtain this goal a change of the mathematics curricula was found to be necessary. One of these changes was that the curriculum, and hence the teaching of the subject should be in accordance with the structure of the subject (Gjone, 1985a; Howson, Keitel & Kilpatrick, 1981; Lerman, 1994; Skovsmose, 1981). This meant that the mathematics curriculum in a way should imitate the logical structure of mathematics, which in this case was a logical structure parallel to the system constructed by Bourbaki. This is what Freudenthal (1973, p.103) characterised as: the *antididactic inversion*. In 1991 (p. 24) he used the words: *from poorer to richer structures*. From Freudenthal’s point of view this was a wrong direction. He wrote:

> Didactically I have opposed rich to poor mathematical structure. This, however, is not enough. One should not be satisfied by staying within mathematics. The rich structures to be offered should also be sought for outside mathematics, albeit with a mathematical-didactical afterthought. (Freudenthal, 1991, p.29)

This last quotation is also adequate for his book *Didactical Phenomenology of Mathematical Structures* (Freudenthal, 1983). In this volume Freudenthal focus and emphasise on the links or connections between mathematics and the world outside mathematics (Usiskin, 1985).

In the volume, *Didactical Phenomenology of Mathematical Structures,* Freudenthal (1983) gives many explicit examples of what he calls mathematical structures:

- length;
- natural numbers;
- fractions;
- ratio and proportionality;
- negative numbers;
- directed magnitudes;
- functions;
- geometrical structures;
- topological structures
- combinatorial structures.

In chapter 7 in this book he has, with the help of several examples, explained or illustrated what he meant by the term mathematical structures. After presenting these examples he states at page 215 in that it is unusual to define (mathematical) structures explicitly as he has done, and that, most often, it is defined implicitly in the following manner:

> *one introduces*
> a set
> *with*
> certain relations on it
> *and requires these relations to observe*
> certain postulates. (Freudenthal, 1983, pp. 215-216)

He then gives three examples of different mathematical structures: a *group*, a *metric space* and a *topological space*.[5] These three examples are what may be called, traditional examples of mathematical structures. This is particular the case for the mathematical structure called a group, which often is chosen as an example of a mathematical structure (Aubert, 1973; Bell, 1976; Bourbaki, 1950; Christiansen, 1969; Collier's, 1985; Lamon & Scott, 1970; Speranza,

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[5]Freudenthal's definition of these mathematical structures are:

> *A* group *is defined as a set* G *and an operation (relation)* $ab = c$ *on that set, and the postulates, associativity, an identity element, and an inverse for each element.*
> *A* metric space *is a set* R *of 'points' with a distance relation for each pair of points, which satisfies the usual requirements for a distance relation.*
> *A* topological space *is defined as a set of* R *'points' with a relation of 'being close to each other' and some given postulates or requirements, which he (Freudenthal) did not specify.*

1994). It is interesting to notice that Freudenthal has exemplified each of the three mother-structures identified by Bourbaki, see page 32. Compared with the definition of mathematical structures as given by Bell & Machover (1977, p. 9) Freudenthal has given the same definition, even though it is not so formal. An informal interpretation of Freudenthal's definition of mathematical structures as found in the volume *Didactical Phenomenology of Mathematical Structures*, is: A mathematical structure consists of a set and at least one relation on that set, and these relations (or eventually the relation) are not chosen by random, they have to meet some given postulates or axioms. [6] According to the nature of the relation(s); i.e. which postulates or axioms they satisfy, different types of mathematical structures are defined. It is noticed that each of the three examples of mathematical structures given by Freudenthal in this context belongs to the 'body' of pure mathematics. Looking at the explicit examples given and elaborated ahead of the definition, a question arises: Why did not Freudenthal, in this context, exemplify mathematical structures with some of his previous given explicit examples?

A closer look at the examples of mathematical structures which Freudenthal (1983) has dissected, analysed and explored (Usiskin, 1985), shows:

- The first example of a mathematical structure is *length*. This is an explicit example of a metric space; i.e. an order structure.
- Natural numbers and fractions (rational numbers) with various operations exemplify groups; i.e. algebraic structures.
- Graphs are examples of topological structures.

Why did not Freudenthal use some of these as examples of mathematical structures in connection with the definition? Is the answer so simple; he found it is obvious that his presented examples are mathematical structures, and therefore there is no need for further explanation? Or, is this an indication that he has used the notion, mathematical structures, in two-fold manner, a non-formal and a formal? Do we have or find a parallel to what Corry (1992) maintains for Bourbaki (see page 33)? Another indication for using the notion in both a formal and a non-formal manner is found in what he wrote in 1983, "Without much ado I have used the word "structure" many times. I will explain it now more systematically." (Freudenthal, 1983, p. 210).

Anyhow, the answer to the formal/non-formal question is not important for this work, and since answering the question is a demanding task, it will be out of the scope for this work.

[6] A more formal interpretation of Freudenthal's definition is the following:
A *relation* is a *set* (collection) of *n*-tuples (a *cartesian subset*). The phrase "require these relations to observe certain postulates" (p. 216), means that there had to be some *operations* (relationships) between the components of (to) the *n*-tuples, and that these operations have to meet some given axioms or postulates.

In his last major work, *Revisiting Mathematics Education. China Lectures*, (Freudenthal, 1991), Freudenthal also discusses mathematical structures. Section 1.2 of this volume is titled *Structure and structures*. There he writes:

> In mathematics the relationship between form and content is reflected by that between something having or being a structure. Structuring is a means of organising phenomena, physical and mathematical, and even mathematics as a whole. (...)
> A few examples will give us a clearer view of what structure means in mathematics. (Freudenthal, 1991, p. 20)

He then gives the following explicit examples of “what structures means in mathematics.” (p. 20):

- combinatorial (a tetrahedron with relations between vertices, edges and faces);
- geometric (a tetrahedron considered as a solid space with relations as distance, area, volume etc.);
- topological (deformation of a tetrahedron with relations which he find “less easy” (p. 21) to explain);
- algebraic (whole numbers with relations as order, addition and maybe multiplication (multiplication may be dependant of addition), rational numbers, real numbers, complex numbers with operations as addition and multiplication).

All these examples have some common properties: There is a collection of something (a set) and at least one relation between the members of this collection (elements of this set), in addition the relation(s) has (have) to satisfy some specified requirements. This is identical with the interpretation given at page 35 of his definition as found in *Didactical Phenomenology of Mathematical Structures*. This is not surprising, for according to Freudenthal “…the present book (*Revisiting Mathematics Education*) adds nothing but itself to work that I have published in the past in various places.” (Freudenthal, 1991, p. xi). The difference between this volume, *Revisiting Mathematics Education,* and *Didactical Phenomenology of Mathematical Structures,* is that Freudenthal in the former publication gives a more concise and balanced description (representation) of the relation between “illustrative examples and theoretical reflection” (Keitel, 1993, p. 164). The latter shows however a multiplicity of concrete details and the richness of mathematical content which can be found in relative simple examples.

2.3.2 Other mathematical educators

At the *First International Congress on Mathematical Education*, ICME-1, Lyon 1969, Bent Christiansen in his plenary address stated “The foremost goal of mathematics on scientific level is *the study of structures*.” (Christiansen, 1969, p. 139). For mathematics education understanding of the subject matter in the curriculum is an important objective. In order to attain this objective Christiansen, in this address, asserted that at an intermediate level "unifying concepts and structures" (p. 146) played a decisive role. This implied in turn that it was necessary to use "… the language of sets, relations and functions, as well as the use of structures like groups and vector spaces" (p. 146). In his address Christiansen did not give what Freudenthal called an implicit definition or an explanation of the term mathematical structures. He exemplified by using well-known examples of mathematical structures: "groups, rings, fields, topological spaces, metric spaces, vectorspaces, etc. etc." (p. 148).

In a later paper, (Christiansen, 1990a), where he discussed the traditional teaching model in mathematics, the 'broadcasting-model', he touched on the term mathematical structures. In this context the term was explained as "the … structures of objectified mathematics" (p. 4). The same year, in another paper (Christiansen, 1990b), he used the phrase "the axiomatic construction of a structure" (p. 31) which gives what may be called an indirect definition of a mathematical structure. This because an axiomatic construction presupposes at least one set and at least one relation, which satisfies some specified postulates (axioms) between the elements of the given set(s). This is the same definition as Freudenthal gives in *Didactical Phenomenology of Mathematical Structures*, see page 35.

The main question in a study by Lamon and Scott (1970) was "can young children be expected to gain an appreciation of mathematical structure?" (p. 95). In their view a structure had to include "some sense of a deductive algebraic system or some feeling for group theory." (p. 95), which can not be classified as a strict definition. In their research *isomorphism* was selected as the mathematical structure to investigate, where the *Klein group* was used as the underlying model.

It is emphasised by Steen (1990) that mathematics is much more than arithmetic, measurement, algebra and geometry, which has been the traditional context of school mathematics. He gave a list consisting of seven elements (ideas) missing in the traditional school mathematics. The first member of this list was "specific mathematical structures" (p. 3) which he exemplified as numbers, algorithms, ratios, shapes, functions and data.

Some educators have used the term mathematical structure or mathematical structures without any further explanation, (Schoenfeld, 1992), while others may give examples or a very 'weak' definition. For example Bell (1976) explains mathematical structures as "inter-related systems of relational concepts." (p. 2.6). He then exemplifes inter alia with the group S_3, the rational numbers, the functions $y = kx^2$. Another example is Dreyfus (1991) who asserts that modelling; i.e. "…finding a mathematical representation for a non-mathematical object or process." (p. 34), is to construct a mathematical structure or theory; i.e. a mathematical structure is juxtaposed with a mathematical theory. This is also asserted by Van Dormolen (1986). Algorithms, functions and ratios are examples mentioned by Volmink (1994, p. 60).

Kieran (1992) in her article says that algebra deals with the symbolisation of "general numerical relationships and mathematical structures and with operating on those structures." (p. 391). She exemplifies with real and complex numbers, equations, polynomial and rational expressions, functions, sequences and series.

Mamona-Downs & Downs (2002, p. 179) maintain that "… how practitioners of mathematics (both the learner and the expert) mentally interact with mathematical structure" is an important factor of advanced mathematical thinking. This mental activity is called *Reflection on Mathematical Structure* (RMS), and it is described as "… conscious mental response to the form in which constructs (objects, expressions, procedures, proofs, etc.) are presented mathematically." (p. 179). This means that the term mathematical structure more or less is used as a synonymous for a mathematical construct.

In a paper presented at PME 28, Dreyfus & Hoch (2004) argue that the traditional definition of structure, in the meaning mathematical structure, is not a helpful definition in an algebraic high school context. As a consequence of that stand they gave, in another paper they presented at the same conference, this definition "*Any algebraic expression or sentence represents an algebraic structure*." (Hoch & Dreyfus, 2004, p. 3–50).

A French group of researchers, with G. Vergnaud in front, has in several articles focused on what they called *additive structures* and *multiplicative structures*. These notions are explained as sets of situations involving addition/subtraction and multiplication/division (Vergnaud, 1988).

These examples reveal that the term mathematical structure(s) are frequently used in mathematics education, but also that the researchers in mathematics education have not agreed on the understanding or the definition of the term. As put by Hoch & Dreyfus (2004, p. 3–50) "It [the term] is used in the field of mathematics education to cover a various different meanings."

2.4 Definition mathematical structure

On the background of the preceding the question is: Is it, everything taken into consideration, possible to give a definition of the term (concept) mathematical structure?

There are some requirements a mathematical structure should meet. Since it is indisputable that variables are needed in mathematics it must be possible to handle variables and/or constants in a mathematical structure. Another requirement is that it should be possible to construct or make expressions consisting of variables and/or constants, and thirdly it should be possible to make statements. This leads to the following definition a mathematical structure:

A mathematical structure consists of

1. a set

2. one or more operations
3. one or more relations

This definition does not differ particularly from the definitions found in the literature; see for example the definitions to Freudenthal (1983) and Bell & Machover (1977).

The set ensures the existence of variables and/or constants. The operation(s) make it possible to construct expressions, and the relation(s) implies that statements can be made. For example the natural numbers, the operations of addition and multiplication, and the identity relation constitute a mathematical structure.

2.5 Examples of mathematical structures

2.5.1 Function structures

A formal definition of a function:

> A function *f* is a *unique mapping* from one set to another set; i.e. a mapping from a set *A* to a set *B* that to an element in *A* assigns a unique element in *B*.

Let *f* be a function from *A* to *B*; i.e. $f : A \to B$, defined by $f : x \mapsto y$ where $x \in A$ and $y \in B$ then the function can be described by $f = \{(x, y) \mid x \in A, y \in B\}$.

Functions can be collected or organised in sets, for example the set of linear functions, the set of polynomial functions, the set of continuous functions, the set of trigonometric functions, etc… On these sets various operations can be defined; i.e. addition, subtraction, multiplication, division, composition of functions. The identity relation will be one relation defined on those sets. This implies that if *f* and *g* are two functions the expressions $f + g$, $f - g$, $f \cdot g$, $\frac{f}{g}$, and $f \circ g$ can be calculated. Functions are therefore a mathematical structure.

2.5.2 Number pattern structures

A number pattern that frequently is found in mathematical textbooks is the *triangular numbers*. As is well known these numbers can be defined by:

$$T_1 = 1,\ T_2 = 1 + 2,\ T_3 = 1 + 2 + 3,\ \ldots,\ T_n = 1 + 2 + 3 + \ldots + n.$$

The triangular numbers together with operations as addition and multiplication, and the identity relation is a mathematical structure.

The triangular numbers can, however, be defined in at least two additional different manners:

> With the explicit formula $T_n = \frac{n(n+1)}{2}$ where $n = 1, 2, 3, \ldots$,
>
> or with a recursive formula $T_1 = 1$ and $T_n = T_{n-1} + n$ for $n > 1$.

A recursive formula defined on the positive integers is a linear combination of values with lower index than *n* and a value depending on *n*; i.e. a formula of the form $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r} = f(n)$ where the c_i 's is constants. Such a formula is also called a linear recurrence relation. If $f(n) = 0$ for all $n \in \mathbb{N}$ the relation is called *homogenous*, otherwise it is called *nonhomogenous* (Grimaldi, 1994). For the case of triangular numbers the recursive formula is a nonhomogenus first-order linear recurrence relation.

Since both rules for defining T_n are single-valued, and having in mind section 2.5.1, the triangular numbers can be viewed under the perspective of functions.

Another famous number pattern is the *Fibonacci numbers.* As will be known these numbers are 1, 1, 2, 3, 5, 8, 13, 21,… . Usually these numbers are defined by the recursive formula: $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, which is a homogenous second-order linear recurrence relation. As for triangular numbers, these numbers together with the operations and relation, inherited from the natural numbers, is a mathematical structure.

2.5.3 Equivalence relations

A traditional and important example of a mathematical structure is *equivalence relations*. As is well known an equivalence relation is a relation *R* on a set *S* which satisfy the following laws:

Reflexive: aRa for all $a \in S$

Symmetric: $aRb \Rightarrow bRa$ for all $a, b \in S$

Transitive: aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in S$

Examples of equivalence relations are: The relation '=' and the rational numbers, the relation 'congruent modulo *n*' ($\equiv (\bmod\, n)$) and the integers, the relation 'parallel lines' in the plane, and the relation of congruence between triangles in the plane.

A property with an equivalence relation *R* defined on a set *S* is that the relation *R* determines a decomposition of *S* into mutually disjoints non-empty subsets, *equivalence classes*. Each equivalence class consists of mutually equivalent elements, elements of different equivalence classes being non-equivalent.

3. THE TASKS

As mentioned in the preceding chapters, the pupils' texts which constitute the data corpus for this research, are solutions to some of the tasks used in the Tangenten competition. These tasks are therefore an essential premise supplier for this study, consequently it is required to raise some questions concerning the selection of the tasks. This selection rests on ideas from both mathematics and pedagogic. The overall question for this chapter is:

What was the reason for using these tasks?

The construction of tasks rests on two cornerstones, which are:

1. a personal pedagogical basis;
2. a pedagogical philosophy.

These two cornerstones will be the focus for the first part of this chapter, the last part of the chapter will concentrate on the classification of the tasks related to the literature.

3.1 A personal pedagogical basis

The personal history outlined in the preliminary part, was the pedagogical background when the request for being responsible for the Tangenten competition was addressed. The outcome of using open-ended tasks in the teaching of college students, the mainly positive experiences gained was therefore a contributory cause to the decision of using the same type of tasks in the competition. The difference in age between the pupils, the participants in the competition, and the students at the college, was not considered to be of any importance considering the type of tasks. This difference was however important when an idea for a task was sorted out, and will be discussed in section 3.2.1.

3.2 A pedagogical philosophy

Even though Mellin-Olsen not was responsible for the competition tasks, he had, what may be called, a very strong and indirect influence through both the competition rules, and the given oral guidelines concerning the tasks.

The first version of the competition rules was proposed by Mellin-Olsen, and on the basis of these rules and personal information from Mellin-Olsen the researcher accepted to take the responsibility for the competition section in Tangenten.

3.2.1 Competition rules

A competition demands rules which take care of different aspects of the competition. For this particular competition the aspects could be identified as:

- *practical*: who can participate in the competition, and how to behave in the competition; i.e. what was the pupils allowed to do, and the role of the teacher;
- *economical*: the number of prizes, and the amount of the prizes;
- *evaluative*: who evaluate the solutions, and against what standards will this evaluation be worked out.

In addition information about time limits, how and where to send the solutions, had to be announced.

The first solutions revealed that it was necessary to amplify the competition rules. This lead to the first revision of the competition rules, later on it became necessary to carry through further revisions. The competition rules and the revisions of the rules are reviewed in appendix A.

The practical aspect of the rules consisted of (sub)aspects which were of great importance for both the selection of tasks, the data collected, and the analysis of these data. The aspect concerning the selection of the tasks is discussed in this chapter, the aspects concerning the collected data will be handled in chapter 4, and they are also commented on in appendix A. The analysis of these data is done in chapters 5-10.

The economical and evaluative aspects of the competition rules are of little significance for this study, and they will not be discussed beyond what is done in appendix A.

Even though the main objective of the competition rules was to inform the competitors about the competition, and clarify the basis of the competition, the competition rules gave at the same time norms for the construction of the tasks. The three main rules stated, see appendix A, section 12.2:

1. The competition was for classes, not individuals;
2. It was competed in three different groups, grade 1-3, grade 4-6 and grade 7-9;
3. The teacher was only allowed to interpret the text of the task for the pupils.

As mentioned in section 1.1 the 1997 school reform in Norway implied that the children should start their schooling at age six whilst they started at age seven before this reform. One consequence was that the length of the compulsory education in Norway was increased from 9 to 10 years. This work uses the old numbering of the grades, which mean that a pupil in grade 1 was 7 years old.

The first rule forced the pupils to collaborate on solving a task, and as a consequence it was supposed that the group of pupils, the class as a whole, could handle more comprehensive tasks than individually working pupils.

The second rule gave strong guidelines for what may be called the technical standards to the tasks. According to that rule, and the personal information from Mellin-Olsen, it was one task for all grades, which meant it should also be possible for the pupils from the lowest grades to work on a task. This would most likely imply that solutions could be on different levels. Concerning the construction of the tasks, this rule influenced primarily on two fields, the numbers which could be used in the tasks, and the arithmetic operations performed on those numbers.

A standard organising of the mathematical syllabus in Norwegian compulsory schools, at the time the competition was running, was that grade 1 pupils in their work with numbers, would only be working with the smallest of the natural numbers. This meant that grade 1 pupils usually would be familiar only with the smallest of the natural numbers. For the arithmetic operations in question, it was supposed that addition and subtraction to a certain extent would be familiar also for the lowest grades, while that would not be the case for multiplication and division. It should therefore be possible to work on and solve a task without using those two last operations.

Seven of the eight tasks constructed for the competition satisfies both the number restriction and the operation restriction. One task, *The Hundred Square*, has multiplication as a vital operation.

Since the target group for the competition was pupils from grade 1 to grade 9, the reading abilities to the pupils would be at various levels. Their reading abilities, or lack of such, could then be an obstacle for their participation in the competition. In order to overcome this obstacle, the third rule allowed the teacher to interpret and/or communicate the task for the pupils. This rule restricted the role the teacher could play in the solving process. In volume 2 number 1 of the journal, this rule was strengthen or emphasised by rule number 4. See appendix A, sections 12.3.3, 12.4.1, 12.5.1 and 12.6.1. In the context of the competition the teacher was only permitted to interpret (or translate) the text of the tasks for the pupils. This rule made the 'linguistic construction' of a task less challenging. It was, in a way, sufficient to assure that the text of a task was understood by the teacher, then the teacher could do the interpretation for the class. That does not mean that the task setting was unimportant, see chapter 4.2.2.

In addition to the practical, economical, and evaluative aspects, the rules also reflected important aspects of the underlying *pedagogical philosophy* of the competition. This philosophy was not affected by the revisions of the rules. The main pedagogical philosophical aspects of the competition could be identified by:

1. *collaboration*;
2. *communication*.

It is noticed that these two pedagogical philosophical aspects, collaboration and communication are central in Mellin-Olsen's philosophy on mathematics education and mathematics teaching (Mellin-Olsen, 1987,1993a). The rules which primarily reflects these aspects are the first, third and fourth rule, see appendix A.

Collaboration between pupils

One of the main underlying philosophies for the competition was to encourage collaboration or co-operation between pupils; i.e. that pupils should collaborate on solving mathematical tasks. This was not a new idea neither was it an original idea, because since the 1970's the aspect of collaboration in mathematics education had been prevalent in many European countries (Mellin-Olsen, 1993b). In 1990 it was also one of the objectives in the mathematics curriculum *M87* ((The) Ministry of Education and Research, 1987, 1990). This objective was not a new objective in *M87*, it was also a part of the objectives in *M74*, even though the collaboration aspect in *M74* was not emphasised as in *M87*.

Despite the fact that collaboration in mathematics education was one of the objectives in the curriculum, and had been that for several years, it was assumed that organised collaboration in the mathematics lessons, in the Norwegian compulsory schools, were relatively unusual at the time the competition started, September 1990. However, as far as known there had, until September 1990, been no research dealing with the collaboration between the pupils in the mathematics lessons in the Norwegian schools. It was reasonable to suppose that some teachers frequently used group-work in their mathematics lessons, but the extend of group-work in mathematics was not known. On the other hand it was also supposed that the majority of the mathematics teachers did not use group-work at all or only used it sporadic. In the case of no or little organised collaboration, it can not be concluded that collaboration did not occur. In many cases the pupils will collaborate in an informal or not organised way. The pupils talk with each other, they compare their answers or ask some of their classmates for a hint, a plan or a line of action.

The main reasons for the competition's request on collaboration between the pupils were:

- collaboration in solving problems is an important aspect of mathematics;
- collaboration presupposes communication;
- collaborating pupils would most likely handle more comprehensive tasks than pupils working individually.

The importance of collaboration in mathematics is stressed by both mathematicians and mathematics educators. In the history of mathematics there are some famous examples of collaborating mathematicians. Bertrand Russell and Alfred North Whitehead collaborated on *Principia Mathematica* (Newman, 1956; "Russel, Bertrand", 2002; Struik, 1967).[7] Other examples are the collaboration between Godfrey Harold Hardy and John E. Littlewood ("Hardy, Godfrey Harold", 2002), and the collaboration between Sinirsva Ramanujan and Godfrey Harold Hardy (Kanigel, 1991). It is also appropriate to mention the collaboration, over many years, by the members of the Bourbaki-group. Borel (1998) emphasise that it had

[7] A reference of the type ("Name", Year) refer to an internet address, APA style 77, 'Stand-alone document, no author identified, no date'.

not been possible to achieve the result of the Bourbaki-group without collaboration. The Hungarian mathematician Paul Erdös, who died in 1996, is known for his productivity, and his collaboration with mathematicians all over the world (Hoffman, 1998). The mathematicians Peter Hilton (Steen & Alexanderson, 1985), and Persi Diaconis (Albers, 1985) emphasised that collaborating is more efficient then working alone, in addition Hilton also stressed that collaborating is enjoying. Another and recent example is the collaboration between Michael Atiyah and Isadore Singer which were awarded the 2004 Abel Prize (Raussen & Skau, 2004). The importance that mathematicians attach to collaboration is also stressed by Burton (1999a, 1999b). Her research revealed that "Mathematics is a team sport" (1999b, p. 36), and that mathematicians "who claimed only to do individual work was extremely rare" (1999a, p. 127). Looking at mathematical journals it is observed that it is no exception that mathematicians publishes joint papers.

The collaboration aspect is also emphasised by many mathematics educators, and by researchers in mathematics education. According to Hoyles "mathematics educators have turned their attention to collaborative modes of learning and to the potential role of discussion in mathematics classrooms" (Hoyles, 1985, p. 205). This change of attention towards collaborative methods in mathematics education is also emphasised by Schoenfeld (1992), and by Niss who wrote that collaboration seemed to be one of the aims, for primary and lower secondary mathematics teaching, "that the vast majority of countries in the world wish to pursue" (Niss, 1996, p. 32).

The importance that both mathematicians and mathematics educators have assigned to collaboration, is in sharp contrast with one of the beliefs on the nature of mathematics that students have acquired (Schoenfeld, 1992). The picture of mathematics "as a solitary activity, done by individuals in isolation" (Schoenfeld, 1992, p. 359) is also mentioned by Fosse (1995, 2004). She made in the beginning of the nineties video recordings of Norwegian children in kindergarten acting (playing) mathematics lessons at school. The video recordings shows that these kids already had acquired the metacognition that mathematics is a solitary activity, and it should be done in isolation and in complete silence. No collaboration was allowed, this in sharp contrast with a similar lesson when the kids were acting a lesson in the Norwegian language.

The stress which has been laid on collaboration in mathematics education, has resulted in a huge number of books and articles, and it would be an arduous task to try to give a complete picture of this diversity of publications. See for example Alrø & Skovsmose (2002); Christiansen (1990c), Christiansen & Walther (1986), Dekker (1987, 1994), Dekker & Elshout-Mohr (1998), Freudenthal (1973, 1978, 1991), Greeno (1988), Mellin-Olsen (1991) and Resnick (1987). Even though collaboration was an important goal for the competition it is not the focus of this work, therefore the collaboration aspect will not be discussed further.

Communication

Over the last decade there have been a growing interest for communication in mathematics. In 1988 Lynn Arthur Steen, at that time President of the Mathematical Association of America, wrote “communication is an important goal of mathematics instruction.” (Steen, 1988, p. 419). The importance of communication in mathematics has been stressed by many researchers (Ellerton & Clarkson, 1996), which in turn has been reflected in reform programs (Ernest, 1989). In *Standards* (National Council of Teachers of Mathematics, 1989) communication is one of the standards labelled in each of the sections. The 1996 yearbook from NCTM, *Communication in Mathematic*s *K-12 and Beyond* (Elliott & Kenny, 1996), contains 28 articles, each dealing with different aspects of communication and mathematics. Different aspects are also reflected by researchers in mathematics education. For example Mellin-Olsen (1987) stresses the importance of communication as a tool to share ideas, develop strategies, and to carry out projects. This aspect of communication is also emphasised by many researchers in mathematics education; e.g. Christiansen (1990c), Dekker (1987, 1994), Dekker & Elshout-Mohr (1998), Freudenthal (1991) and Senn-Fennell (1995). Communication as a tool for promoting reflection is another aspect which is mentioned by Freudenthal (1991) and Wistedt (1994). A third example is the language of technology and mathematics. During the past decades this language has invaded many areas of everyday life in our society (Keitel, 1998; Nissen, 1990, 1994; Senn-Fennell, 1995; Skovsmose, 1994a, 1994b).

Niss (1994) and Keitel (1998) contrast the presence of mathematics in large parts of our surroundings, and that this mathematics is not always visible. This mathematics is often hidden, but it is still present, which means that mathematics plays an important role in our society. In order to prepare the pupils for living as active and responsible citizens in this society, and stimulate a democratic development it is necessary that they can understand and communicate this language (Blomhøj, 1994; Elliott, 1996; Niss, 1990, 1996; Nissen, 1994). The communication aspect is also stressed in the *M87* curriculum for compulsory education in Norway ((The) Ministry of Education and Research ,1987), in the *L97* curriculum ((The) Ministry of Education, Research and Church Affairs, 1996) and when this is written in the latest (and at this time provisional) curriculum, *Kunnskapsløftet* (Utdannings- og forskningsdepartementet, 2005). Communication in mathematics has thus a democratic aspect. The democratic aspect implies that the pupils should be invited to ‘speak mathematics’ and ‘listen to mathematics’; i.e. that they should be invited to express mathematical concepts and results using their own words, and also listen when classmates expresses their understanding of the mathematics involved. The pupils should attain both an active and a passive communicative competence (Senn-Fennell, 1995), which is strongly emphasised in *Kunnskapsløftet* (Utdannings- og forskningsdepartementet, 2005).

The growing importance of communication in mathematics education has, as the case of collaboration, resulted in a huge number of books and articles. See for example Alrø & Skovsmose (2002), Ellerton &Clarkson (96), Elliott & Kennedy (1996), and Steinbring, Bartolini Bussi & Sierpinska (1998). Even though communication was an important goal of the competition, it is, as the case of collaboration, not the focus of this work. The communication aspect will therefore, in this work, not be discussed further.

3.2.2 Guidelines from Mellin-Olsen

In addition to the written competition rules Mellin-Olsen gave personal information and guidelines. It would not be correct to say that these guidelines had the form of an instruction, it would be more correct to characterise these guidelines as strong wishes from Mellin-Olsen. He wished that the tasks should have a broad perspective, which meant that it should be possible to extend the tasks beyond what was asked for in the competition, and if possible, extensions in various directions. With respect to the tasks this was a particularly important guideline.

These guidelines together with the competition rules and a wish to emphasise the underlying pedagogical philosophy of the competition, collaboration and communication, made relatively strong demands upon the construction of the tasks. It was desirable that a task could be solved in different ways; i.e. that there was not one and only one way to solve a task or a task should give room for solutions at different levels. These requirements, or may be more correct wishes, was also in line with the personal pedagogical basis, and they pointed at tasks with an open ending.

The next sections will give a literature review of related to open-ended tasks and classify the tasks used in the competition.

3.3 Open-ended tasks

3.3.1 Introduction

During the 1960's *investigational* activity was introduced in mathematics teaching in Britain (Wiliam, 1993), this was followed up by the Cockroft-report (1982) which emphasised investigations as an important mathematical activity. A detailed overview over the historical development of investigations in mathematics is found in Wiliam (1993). In the period 1971 to 1976 the effectiveness of what was called *open-ended problems* was in Japan developed as a tool to evaluate higher-order-thinking skills in mathematics (Becker & Shimada, 1997). In the same period the *reinvention* perspective was introduced in The Netherlands, this perspective was the forerunner to the *realistic mathematics* perspective, (Freudenthal, 1973; Gravemeijer, 1994; Treffers, 1991,1987; van den Heuvel-Panhuizen, 1998).

The use of investigational tasks or open problems in mathematics education was in the 1980's widely spread and the research on the use of such tasks increased (e.g. Becker & Shimada, 1997; Boaler, 1998; Nohda, 1991, 1995; Pehkonen, 1995a, 1995b, 1997a; Silver, 1995; Stacey, 1995; van den Heuvel-Panhuizen, 1998; Wiliam, 1993).

Silver (1995) points out that the term open problem has several different meanings. In mathematics it is used for a problem that has remained unsolved; i.e. *Goldbarch's conjecture*, the zeroes to the *Riemann Zeta-function* or a formula for prime numbers. Until the beginning of the 1990's *Fermat's last theorem* was an open problem. In mathematics a problem is open until an acceptable solution has been given. Acceptable means here that the majority of mathematicians agrees on that the solution satisfies the necessary criteria to be accepted as a correct solution. That does not mean that *this* solution will be accepted by mathematicians in the future. This acceptance process is illustrated by Lakatos (1993) in his book *Proofs and Refutations*. In this book he used the historical development of the proof for *Eulers's polyhedral theorem* as an example.

On the other hand, in mathematics education the meaning of the term open problem is not unique. A problem which give cause for different interpretations or to different answers which are acceptable, is characterised as open. This is also the case if a problem "invite different methods for solution." (Silver, 1995, p. 68) or "if they naturally suggest problems and generalisations." (Silver, 1995, p. 68).

From the literature it appears that different notions linked to the openness are used, many writers have used the notion open-ended *problem* (e.g. Nohda, 1991, 1995; Pehkonen, 1995a, 1995b, 1997a; Shimada, 1997; Stacey, 1995), while Wiliam (1993) used the notion open-ended *task*, and Boaler (1998) used open-ended *work*. Since this study is not directly linked to or grounded in the theories about problem-solving, and since the notion used by Boaler was found to be to wide it was decided to use the notion open-ended task(s) in this work.

Nohda (1995) distinguish between, teaching using 'open-ended problems', where the term, open-ended problems, is in brackets, and open-ended problems, the term is not in brackets. He defines teaching using 'open-ended problems' "as an instruction in which the activities of the interaction between mathematics and students promote varied problem solving approaches." (Nohda, 1995, p. 58), and this teaching make use "of non-routine problems: problem situations, open-ended problems and process problems." (Nohda, 1995, p. 59). He does not indicate the distinction between those three types of problems.

3.3.2 The concept open tasks

In order to answer the question:

What is an open task?

some researchers prefer to answer what may be called the complementary question; i.e.

What is a closed task?

and then consider all tasks which are not closed, as open and/or open-ended (Pehkonen, 1995a, 1997b; Shimada, 1997).

A task has both a starting situation and a goal situation. If a situation can be exactly explained the situation is called closed, if not the situation is called open. The following table can then be made:

Task type	Starting situation	Goal situation
A	Closed	Closed
B	Closed	Open
C	Open	Closed
D	Open	Open

Table 3.1 Different types of tasks

It is usual to characterise a task as closed if both the starting situation and the goal situation is closed (Pehkonen, 1995a; 1997b). Shimada (1997) has used a similar definition for a closed problem or as he also called it a *complete* problem. An open task is then a task where the starting situation and/or the goal situation is open. This implies three possible different types of open tasks: B, C, and D. Based on the personal pedagogical basis and the pedagogical philosophy, outlined in sections 3.1 and 3.2, tasks of type A, the closed one, was not considered to be relevant for the Tangenten competition, they are therefore of little or no interest for this study.

Silver (1995) used the term open problems. He has given four categories of open problems. A problem can be open if it:

1. opens up for different interpretations;
2. opens up for different answers;
3. invites to different methods of solution;
4. suggests other problems or generalisations.

Silver's categorisation does not directly concur with the types B, C, and D from table 3.1, but there are relationships or links between these two categorisation sets. If the starting situation is open it could give opportunity to different interpretations, and hence also too different acceptable answers. The opposite is also true, if a task gives rise to different interpretations, the starting situation has to be open and not closed. If a task gives different acceptable answers the starting situation could be either closed or open, for example a linear diophantine equation can give several different acceptable answers.

A task or problem could have a closed starting situation and still invites to different methods of solution; i.e. the goal situation is open. On the other hand, if a task invites to different methods of solution the starting situation could be either open or closed. A starting situation, which is either open or closed, could give opportunity to suggest new tasks or generalisations; i.e. the goal situation is open, and if a solution of a task invite to generalisations or suggestions of new tasks the starting situation could be either closed or open. It can therefore be concluded that Silver (1995) has used the term open problem for a problem where the starting situation and/or the goal situation is open, the types B, C, and D.

The concept open-ended tasks

What are then an open-ended task? Of the three types of open tasks, B, C, and D from table 3.1, Pehkonen (1995a, 1997b) concluded, without any explanation or argumentation, that it is only type B, closed starting situation and open goal situation, which is the open-ended problem/task. This conclusion is found a bit surprising, but by reason of the lack of argumentation it is impossible to figure out why he did not include tasks where both starting and goal situation are open, type D, as open-ended. Further on, his use of the term 'open-ended problem' is unclear or problematic. The reason is that he stated explicitly that task of type B "is the open-ended problem" (Pehkonen, 1995a, p. 55; 1997b, p.8), and later on he has listed open-ended problems as one out of five examples or categories of open-ended problems of type B, these examples are: Open-ended problems, real-life situations, investigations, problem fields and problem variations (Pehkonen, 1995a, p. 56; 1997b, p.9). At the same time the term, open-ended problems, is used as a name for a type of tasks, a category, and a name for a particular member or object belonging to this category, and the difference between the member of the category open-ended problems is not indicated; i.e. the difference between open-ended problems, real-life situations, investigations, problem fields and problem variations.

Wiliam (1993) emphasised that an investigation has two major features, openness and purity. He defined openness of a task by:

> Task A is more open than task B if:
> - all the acceptable interpretations of task B are acceptable interpretations of task A, and
> - there are acceptable interpretations of A that are not acceptable interpretations of B.
>
> (William, 1993, p. 10)

In this respect the term *open task* means a task with an open starting situation, it says nothing about the goal situation. For open-ended tasks he defined, quite analogue:

> Task A is more open-ended than task B if:
> - all the acceptable solutions of task B are acceptable solutions of task A, and
> - there are acceptable solutions of A that are not acceptable solutions of B.
>
> (William, 1993, p. 12)

These two definitions is then used to classify tasks. The possibility that a task has an open starting situation and a closed (not open-ended), goal situation is discounted, since it is most unlikely that a task which can be interpreted in several ways should give similar solutions. If both the starting and the goal situation are closed, the task is classified as a problem. This can be summarised in the table:

Task type	Starting situation	Goal situation	Wiliam (1993)
A	Closed	Closed	Problem
B	Closed	Open	Open-ended task
C	Open	Closed	-------
D	Open	Open	Open task

Table 3.2 Wiliam's classification of tasks

This table is just an instrument for classification of tasks. It should not be interpreted that some types of tasks are more superior than other types.

Purity was the other feature used by Wiliam for characterising investigations. By this he meant "that they are located in an essentially 'pure' context. In other words, the 'referent' of the task – what it is 'about' – is within the domain of mathematics." (Wiliam, 1993, p. 12). In other words, even though the context of the task pretend to be outside the world of pure mathematics, the solution is within the world of mathematics.

Another categorisation of task was done in the *AMI*-project, *Applying Mathematics International*, with the participating countries Denmark, England, Germany and The Netherlands. The *AMI*-project operated also with four different types or categories of tasks: *Word problems*, *Problem solving*, *Investigations* and *Mathematisation* (van den Heuvel-Panhuizen & van Rooijen, 1997). Each of these types are defined or characterised by a set of characteristic qualities, and for each type an example is given. Related to the starting situation and goal situation the following characteristics are found (van den Heuvel-Panhuizen & van Rooijen, 1997)[8]:

Word problems:

- Operations can in most cases be deduced from the text;
- Nearly always one and only one good answer.

Example: Sporting club Atilla had 1st of January one year 234 members. In this year 58 members left the club, and 113 new became members.
How many members had this sporting club with the end of the year?

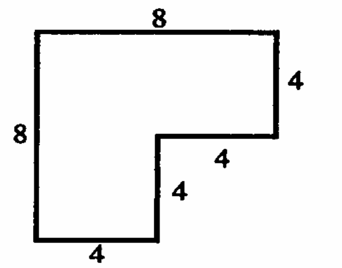
[8] These characteristics and the tasks have been translated from Dutch with assistance of Mr. Ronald Nolet, Volda/Halden.

Problem solving:

- The solving strategies can not be deduced directly from the problem text;
- Leads in many cases to a trial-and-error attempt;
- The solution has a all-or-none character, semi good solutions does in most cases not exist;
- In most cases always one good solution.

Example: Next to is a sketch of a land to farmer Jenkins. He wants to divide this land into four parts which are alike. Each part has identical form and surface.

Find out how this can be done.



Investigations:

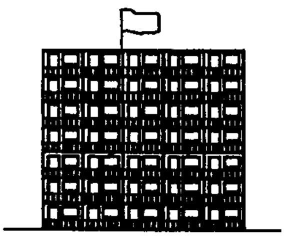
- In most cases descriptive and directed towards a goal;
- In many cases unlimited open (the Dutch text: onbegrensd open);
- Often one have to choose descriptive goals; i.e. more than one correct solution is possible;
- Often it is not possible to decide what is a good solution.

Example: Find out the approximate distance covered by your walking during one day.

Mathematisations:

- In many cases it is a question of adapting personal goals;
- The given information has in many cases to be organised and/or completed and/or omitted;
- Often it is possible with more than one good solution;
- In most cases it is not possible to tell what is a good solution;
- The operations can not be deduced direct from the problem text.

Example: How big is this flag in reality?



Since some of the characteristics is used for more than one category it should be noticed that there is no clear distinction between the categories.

The links or connections between the notions in table 3.1 and the AMI-project is:

Word problems and problem solving can be identified as closed problems. Investigations and mathematisations are open tasks. Investigations are open-ended tasks where the starting situation is closed and the goal situation is open This give the following table:

Task type	Starting situation	Goal situation	AMI
A	Closed	Closed	Word problems, problem solving
B	Closed	Open	Investigations
C	Open	Closed	
D	Open	Open	Mathematisations

Table 3.3 The AMI classification of tasks

As for table 3.2 this table is just an instrument for classification of tasks. It should not be interpreted that some types of tasks are more superior than other types.

A brief summary of the previous indicates that the term open-ended problem/task in mathematics education not has been used in a consistent way. The definition or explanation of the term differ among the members of the community of mathematics educators, there is no unique definition of the concept.

Returning to the question of characteristics of open-ended tasks. In this study the term open-ended task means a task where the goal situation is open; i.e. tasks of the type B or/and type D. In order to separate these two types from each other it is, however, convenient to name each type. Since the terms *open* and *open-ended* are used for categories of tasks, it is found appropriate to use the *AMI*-classification, this implies that a task of type B is called an investigation, and a task of type D a mathematisation. The tasks used in the Tangenten competition can all be classified as investigations.

3.4 Conclusion

Returning to the main question for this chapter:

What was the reason for using these tasks in the Tangenten competition?

Through teaching experience this researcher had revealed that open-ended tasks could have the required quality. The pedagogical philosophy manifested in the competition rules and in guidelines from Mellin-Olsen, pointed also at open-ended tasks as suitable for the competition. However, the decision of using investigations, closed starting situation and open goal situation, as the open-ended tasks was more pragmatic. In the beginning of the 1990's it was supposed, concerning the mathematics classrooms in Norwegian compulsory schools, that the use of mathematisations would be exceptional. For most of the pupils and the teachers this type of task would be unfamiliar. A task with a closed starting situation would, on the other hand, be more familiar to both pupils and teachers. The number of competitors would most likely be higher if the tasks used in the competition were investigations and not mathematisations.

The next chapter will concentrate on the data collection and the analysis method.

4. RESEARCH METHODOLOGY

4.1 Introduction

This chapter addresses methodological issues related to this study. These issues include the areas: The collection of data, the analysis method, the characterisation of the study, the research approach, and the issues of reliability and validity.

4.2 Data collection

As mentioned in chapter 1 the data corpus or the materials that constitute the basis for this scientific investigation, are solutions to some of the tasks used in the *Tangenten* competition. At the time the data was collected it was not the intention that these data later on should be the basis for a scientific research. For a scientific research this is, what may be characterised as an untraditional way to collect data, and it raises some questions which is necessary to discuss. Questions concerning:

- the participation in the competition;
- motives for participating in the competition;
- the teachers;
- the participating classes.

4.2.1 Participation in the competition

The number of classes who participated in the competitions was never large. As a consequence of the low participation the competition came to an end in September 1992. The last competition took place in issue **3**(3). All together there were eight competitions. The number of solutions received for each of the competitions varied from zero to seven, with a total of 25 solutions. The distribution of the pupils' solutions with respect to the grades and issues can be put in the following table:

Year	1990		1991				1992			
Competition / Grade	**1**(1) Sept	**1**(2) Nov	**2**(1) Feb	**2**(2) Apr	**2**(3) Sept	**2**(4) Nov	**3**(1) Feb	**3**(2) Apr	**3**(3) Sept	Sum solutions per grade
1		1								1
2				1						1
3		1	2			1			1	5
4				1		1				2
4-6			1							1
5	1			1	1			1		4
6	1									1
7		3		1						4
7-8		1								1
8				1		1				2
8-9				1						1
9		1				1				2
Sum solutions per issue	2	7	3	6	1	4	0	1	1	25

Table 4.1 Distribution of the pupils' solutions with respect to grade and competition

The notation grade *a-b* means that the class consists of pupils from grade *a* to grade *b* inclusive. The numbering of the grades is the 'old' one; i.e. that was used before the compulsory education in Norway was increased from nine to ten years, see section 1.1.

In 1990 less than 50% of the primary schools (grades 1 - 6) in Norway had separate classes at each of the six grades; i.e. the schools were what is called 6-divided. On the other hand 80% of the pupils attained a 6-divided primary school (Kvalsund, Løvik & Myklebust, 1992). Since 1990 several small schools have been closed down and the percentage of 6-divided primary schools has increased.

In order to refer to the different classes listed in table 4.1 the following co-ordinates are used:

Task in issue **1**(1). *A number and its reverse*.
Class A Grade 5;
Class B Grade 6.

Task in issue **1**(2). *A set of ten cubes*.
Class C Grade 1;
Class D Grade 3;
Class E Grade 7;
Class F Grade 7;
Class G Grade 7;
Class H Grade 7-8;
Class I Grade 9.

Task in issue **2**(1). *The hundred square***.**

Class J Grade 3;
Class K Grade 3;
Class L Grade 4-6.

Task in issue **2**(2). *Black and white squares*.
Class M Grade 2;
Class N Grade 4;
Class O Grade 5;
Class P Grade 7;
Class Q Grade 8;
Class R Grade 8-9.

Task in issue **2**(3). *Arithmogons*.
Class S Grade 5.

Task in issue **2**(4). *In the city*.
Class T Grade 3;
Class U Grade 4;
Class V Grade 8;
Class W Grade 9.

Task in issue **3**(2). *Billiard*.
Class X Grade 5.

Task in issue **3**(3). *Length of trains*.
Class Y Grade 3.

Some of the classes have participated in the competition more than once. With the co-ordinates from above this can be described by:
Class D = Class N;
Class H = Class R;
Class J = Class U = Class X;
Class M = Class T;
Class P = Class V.

The equal sign between two classes does not necessarily mean that these two classes are strictly identical; i.e. that the pupils in the two classes were the same when they worked on the different tasks. There was no information available about the pupils in the classes, pupils can have moved to and from the classes.

In 9 out of the 25 solutions there exist comments from teachers, letters or interviews in *Tangenten*. The comments are on the tasks and/or the way the pupils have responded to and worked on the tasks.

Table 4.1 shows that the number of solution to the different tasks varies considerably from competition to competition. Two of the tasks, those presented in the issues **1**(2) and **2**(2), all together account for more than 50% of the received solutions. No solution was received for the task in issue **3**(1), however, this task is identical with the task in issue **3**(3), and to that task one class responded.

As data corpus for the analysis is used the solutions to the first six tasks in the competition, which means the tasks in volume 1 and 2 of the journal. All together this basis contains 23 solutions, and it is the solutions from Class A to Class W inclusive.

There are probably several reasons for the variations in the number of received solutions. One reason for a class to participate in one of the competitions that is stressed by several of the teachers, was that the pupils found the task interesting or engaging, and therefore they worked with enthusiasm on the solutions; see letters from the teachers to some of the classes (Class B, p. 1; Class L, p. 1; Class M, p. 1; Class P, p. 1 & Class R, p. 1). This has also been stated in an interview with a teacher of one of the participating classes (Høines, 1991). The converse has also been reported, a task did not engage the pupils therefore they did not want to work on the task (Class V, p. 1). In a theoretically perspective this information given by the teachers illustrates nicely the notion *control of knowledge* introduced by Mellin-Olsen (1990, 1993a). If the pupils controlled the *tool level*, the *choice level* and the *goal level* they worked with enthusiasm and were engaged in the solving process. Even though the pupils did not control the goal level when they started to work on the task, it showed however, at a later stage, and for some of the tasks they also took control of this level, it was 'their' task and they wanted to solve it. If they did not take control of this level, the task was not interesting they did not want to work on the task, even though it was possible to win a prize.

A reason for not participating in the competition was that even though the teacher found the task both engaging and interesting, it did not correspond with the syllabus. Therefore the teacher decided that the class could not spend time on solving the task, and consequently they could not participate with a solution in the competition (personal information given by the teacher to Class D and N).

A third reason for these fluctuations in the number of competing solutions could possibly be linked to the publishing months of *Tangenten*. From table 4.1 it is observed that 11 solutions, 44%, have been on a November task, and 7 solutions, 28%, on an April task. This could indicate that it is more likely to let the pupils work with investigational tasks at certain time of the year, but from the data given, it is not possible to draw any definite conclusion concerning the publication months and the number of competing solutions. Even though the question of this variation is interesting, it is not a subject for this research and therefore it will not be discussed further.

4.2.2 The teachers and the solutions

All the solutions were carried out in the pupils' ordinary class context, which mean that no extern researcher was present when the pupils worked on their solutions. Consequently the researcher had no control over the implementation of the working process and the conditions given to the pupils when they carried out their solutions. Against this background and the problem statement the question concerning the *independence* or *trustworthiness* to the data used in this study, becomes important. Independence in this context means that the pupils' really produced the data, and that the pupils had worked without the teachers' intervention; i.e. that the data are the pupils' self-obtained solutions. The question to address can be formulated:

What about the teachers' influence on the pupils' solutions?

From the competition rules it was clear, at least after the first revision, see appendix A section 12.3.3, that the teacher should play a passive role when the pupils were working on a task, a role that may be characterised as the role of a not participating observer. The rules stated that the teacher was allowed to assist the pupils in interpreting the problem-text, but (s)he was not allowed to intervene or involve herself in the pupils' solution process; i.e. provide hints or pose questions that focused the pupils' attention on relevant information or useful ideas.

As mentioned in appendix A section 12.3.3, one solution , Class B, to the task *A number and its reverse* was the teacher's report and summary of the pupils' work. In the perspective of the competition this was not the intention since one of the objectives for the competition was to make visible the mathematics in the pupils' solutions, and not the teacher's interpretation of the pupils' work. A revision or an amplifying of the rules was therefore carried through, it was found necessary to emphasise that the teachers had to play a passive role. The intention behind this revision was to a greatest possible degree reduce the influence of the teacher on the solutions given by the pupils. The solutions should be the pupils' self-obtained product, not the teacher's report and/or summary of the pupils' solution. After the first revision none of the received solutions were a teacher's report or summary of the pupils' work.

One solution to the task *A set of ten cubes,* the solution from Class G, was written with a typewriter. The language used in this solution made it probable that the teacher wrote the solution. This solution was the reason for the second revision of rule 4, see appendix A section 12.4.1. The revision stated again that the teacher had to play a passive role. After this second revision the all the solutions were written by the pupils. The validity to the two solutions mentioned above, those from Class B and from Class G, will be discussed further in the sections with the analysis of the solutions.

Some of the teachers to the participating classes stated that it was very difficult ‘to keep quiet’ when the pupils were working on the tasks Høines (1991) and information given in the letter from the teacher to Class R:

> It was very frustrating for me as a teacher, not to be allowed to intervene and direct (the pupils) to the right track, but that was covered up later on (after the pupils answers was mailed) when we examined the problem. (Class R, p. 1)

This is also one of the conclusions drawn from *The Indiana University's Mathematical Problem Solving Project*, as reported in Lester (1985).

Not being present when the classes worked on their solutions also had a positive aspect. There was no person, unfamiliar for the pupils, a stranger, present in the class. The researcher had therefore not a direct influence on the pupils’ work. The pupils were all the time in their ordinary classroom situation. The classroom situation had not the character of a teaching experiment.

In 1993 when it was decided to use these texts as in a scientific study, it was reflected on interviewing the teachers involved. At that time it was nearly 2½ years since the first competition was held. The decision for not doing the interviews was mainly based on the fact that 2½ years was a relatively long period of time, and the usefulness of such interviews was therefore doubted. The decision of not interviewing these teachers is further supported by another episode. In spring 1994 a letter, addressing some questions concerning a solution to the task *Black and white squares*, was mailed to the teacher of the class. The teacher responded that he was not able to answer the addressed questions owe to the fact that it was nearly three years since the solution was carried out (personal information).

The rules put restrictions on the teacher’s role, but they did not ‘eliminate the teacher’. The teacher was still a vital part of the classroom environment. This means that the teacher obviously has influenced the pupils’ work and their solutions, but to determine the degree of this influence is difficult, and for this particular study is it probably impossible to uncover the teacher’s influence on the solutions. There are at least two areas where this influence is decisive. Firstly, teachers had to present a task for their class and arrange the practical preparation for the pupils’ work. This included among other things, organising the timetable in such a way that their class got the opportunity and enough space of time to work out a solution. The investigational type of tasks used in the competition made it necessary to use some time on the work. In fact the teacher to Class T informed that this class worked on the task, *In the city* for a period of four weeks (Class T, p. 2). Secondly, the teacher could interpret the text of the task for the pupils. For the classes from the lower grades an interpretation was necessary, see section 3.3.1. One teacher, the teacher to Class M and Class T, informed personally that he found the interpretation-rule, rule 3, somewhat ambiguous and confusing. He found it very difficult to know how far he could assist the pupils in the presentation of the task. His dilemma was: What was accepted as interpretation?

From the previous it can be concluded that related to the solving process the teacher was an un-controlled variable. However, this is not the only un-controlled variable. The situation was that the pupils' solutions were worked out under the conditions mentioned above, and little can be concluded about the influence of the teachers. Nor can it be concluded anything about the possible influence from other sources. Homes (parents, brothers and sisters) and friends outside class/school are all vital parts of the pupils' environment. The pupils were not isolated, they were not in a laboratory, but in their ordinary school context, and they worked on a task for a shorter or longer period of time. In that period they had the opportunity to discuss the task outside the classroom. It is impossible to uncover the extent of such discussions. This means that the teacher was not the only uncontrolled variable, parents, brothers, sisters, and friends were also uncontrolled variables.

4.2.3 Motives for solving the tasks

Focusing on the motives for solving the tasks two groups of actors have to be looked upon; the pupils and the teachers. In this context questions to address are:

Why did a teacher wish that her/his class should engage in this kind of activity?

and

Why did the classes solve the task or why did the class engage in this activity; i.e. why did they wish to participate in the competition?

The two next sections will address these questions.

The teachers

In order to answer the first of these two questions it could be of interest to discuss the following:

What categories of teachers were reading Tangenten?

As mentioned in appendix A, section 12.3.2, the intention from the publishers was that the first issue of the journal should be distributed to all the Norwegian primary and secondary schools. There exists, however, no information or knowledge of how many teachers got the opportunity to become acquainted with the journal, but it is reasonable to assume that the readers belonged to a group of teachers especially interested in mathematics. For a large group of Norwegian mathematics teachers the editor, Mellin-Olsen, was well known, and members of this group of teachers would certainly be interested in reading this 'new' journal. It is most unlikely that teachers, not particular interested in mathematics, would spend their time reading a journal of mathematics education.

The teachers, in their letters or in interviews in *Tangenten*, does not give a clear answer to the motive question, but there are some indications for why they let the pupils work with investigations. One of the teachers reflected on her own experience when she studied mathematics. She remembered the 'aha'-experiences, and wanted to give the pupils opportunity to experience some of the same. “I know something about the process which I wanted the children should experience.” (Høines, 1991, p. 12). Another teacher reported in a letter attached to the pupils’ solution to the task *Black and white squares*, that the pupils during their work with the problem observed many relationships which they were unable to formulate in a written language, and he continued:

> (…) the most important is after all to develop the thought and the creative processes and not least to inspire to achievement. This manages the competition in the best possible way. (Class R, p. 1)

The collaboration between the pupils was for some teachers a reason to let the pupils participate in the competitions (Høines, 1991). The collaboration aspect was also supported in a letter from one of the teachers “We choose this problem primarily to practice collaboration.” (Class P, p. 1). Another argument for letting the pupils work with these tasks was the national curriculum for compulsory education in Norway. In the competition period the national curriculum was *M87*, and in the mathematics syllabus problem-solving was a main topic. Compared with former national curricula this subject was a new one, and had no tradition in the Norwegian primary and lower secondary school. In addition the description in the curriculum for that specific subject was what may be called open or vague. The mathematical textbooks written on the basis of the national curriculum implemented this subject in different ways, the number and nature of the problems varied from textbook to textbook. Many teachers were looking for problems from sources outside the mathematical textbooks. For some teachers *Tangenten* become such a source (Høines, 1991).

The pupils

One of the competition rules was that the participating classes could win a prize, an amount of money. From the editor’s point of view this was primarily meant as an ‘artificial’ motivation (‘a carrot’) to get classes interested in starting to work on the tasks. Is the answer to the question of why the pupils wished to participate in the competition the obvious one; they participated by reason of the prizes?

For some of the classes it seems that this was the main motivation, at least when they started to work on a task. Supports for this claim are found in Høines (1991), in a letter from one teacher (Class R, p. 1), and in one solution where a pupil wrote, “We can win 100 NCR” (Class K, p. 6). But there is also indication that even from the very beginning of the solving process the task was the pupils’ motivation. A statement supporting this is found in a letter from the teacher to Class L (p. 1), “The pupils were very enthusiastic in the beginning”. However for some classes another pattern can be identified. It seems that after a period of time, when the pupils had worked on a task for a while, the task itself was the main motivation. The pupils had taken control. Control is here in the way explained by Mellin-Olsen (1990, 1993a), see also section 4.2.1. Support for this assertion is found in letters from teachers. For example the teacher to Class B, wrote in the letter enclosed to the solution of the task *A number and its reverse* “The pupils were so intense engaged with the "mystery"”, and “We should like to carry on further, but now we must return to fractions and per cent again!” (Class B, p. 1). The teacher to Class M gives another example, this time with the task *Black and white squares*, he wrote, “It was a very active group that calculated….” (Class M, p. 1). There is also an indirect support for the statement that the task itself was the main motivation. In one instance it is reported from the teacher to the Class P (and also to Class V) that the class did not find the task *Arithmogons* engaging and therefore they decided not to involve in a solution process to that particular task (Class V, p. 1). The ‘artificial’ motivation, winning a prize was not strong enough, the task did not engage, and thus the pupils did not want to work on it.

4.2.4 The participating classes

Since rule 1 settled that the competition was for classes and not for individual pupils, it is required to have a closer look at the classes. Information or knowledge about the level of learning to the classes, the number of pupils in the classes (class size), and geographical distribution to the classes could be of interest. One consequence of the way the solutions were collected is that, except for the geographical distribution, very little information about the classes is available. Only two teachers informed about their classes level of learning. The teacher to Class V and Class P stressed that the mathematical skill to the pupils in his class was very ‘scattered’. This particular class had only seven pupils, one of the pupils, a boy, was very active in the solving process. In the solution to the task *In the City*, he was the originator for more than half of the solution (Class V, p. 1). The other teacher informed that he looked at his class, which participated twice, as Class M and Class T, as quite average (Personal information).

Concerning the number of pupils in the classes it is likely to suppose that this number varied between 7 and 28. It is known that one class had seven pupils, se above. If some of the classes actually had 28 pupils is not known, but that number was at the time the competition took place the maximum number of pupils that was allowed in a Norwegian class.

The organising of the competition implied that the geographical position to each of the classes is known. The competition rules stated that the competition was open for classes in the Nordic countries, see appendix A section 12.2. However, except for Class Q, which was from Sweden, all the other classes were Norwegian. The Norwegian classes were located in all parts of the country, in both rural and urban areas, however with a majority at the western part of the country.

4.3 Analysis method

The purpose of the analysis method is, as mentioned in chapter 1, the explicit-making of the mathematical structures inherent in the pupils' solutions of some investigational tasks. The only concern for the analysis in this research is the uncovering of mathematical structures and not possible arithmetical errors or inferred cognitive processes. Since the interest is in analysing the produced solutions in terms of mathematical structures, the process can be viewed as a decoding process. A brief description of this method could be 'analysing the pupils' solutions through the glasses of a mathematician'. This method could also be characterised as a decoding process from the language used by the pupils into the formal mathematical language. [9]

A first level of this analysis procedure is illustrated in figure 4.1:

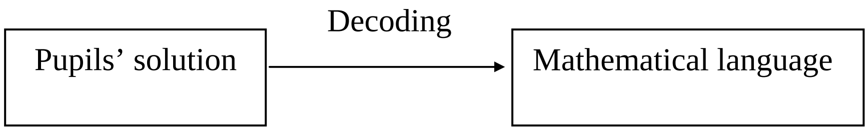


Figure 4.1 Direct decoding process

The term 'mathematical language' means in this context a formal mathematical language; a language that uses mathematical symbols/notations. For some of the solutions or parts of a solution, this decoding process was relatively simple; i.e. the mathematical meaning of the text is clear, and a *direct* decoding is possible. This could be the case if the pupils in their solution already have used mathematical symbols; they use in their solutions what Pimm (1987) calls a *symbolic* style of recording. It could also be the case if the pupils in their solutions used a *mixed* style of recording (Pimm, 1987); few mathematical symbols were used in their solution. A direct decoding can also be utilised if the pupils use a *verbal* style of recording (Pimm, 1987); i.e. very few or no mathematical symbols were used by the pupils in their solutions but the natural language is such that the 'mathematical language' can be easily inferred.

In some cases a direct decoding is not straight ahead or possible, the decoding process gives a product with gaps. This situation is given in figure 4.2. The decoding process leads to an incomplete mathematical description:

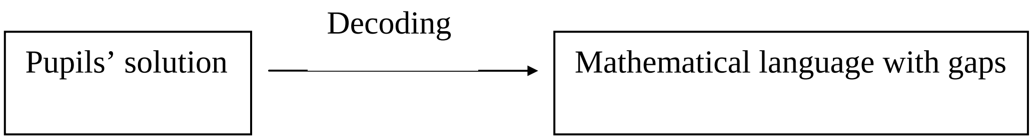


Figure 4.2 Incomplete decoding process

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[9] The following has been discussed by Professor Cyril Julie, UWC and he has contributed to my mind about the decoding process.

This will usually be the situation if a solution is ambiguous; i.e. for the whole solution or parts of the solution there are several decoding possibilities. Missing or inaccurate use of words and/or unexpected use of or combinations of words could cause this ambiguity.

When a group of people is working on a particular problem or task for a period of time, the members of the group obtains specific knowledge about the problem, the results, the solution procedure, and the solution. For the members of the group some of this knowledge may be tacitly understood. A consequence of that could be that the written solution is comprehensible for the members of the group, but could be hard to comprehend or even incomprehensible for a person outside the group, and hence a decoding process is described by figure 4.2.

The goal for a decoding process is presenting a solution in a mathematical language without gaps, as illustrated in figure 4.2. When the decoding process ends with a product dressed in a mathematical language with gaps is it both necessary and required to 'fill the gaps' in order to get a solution written in a mathematical language. In doing so an *emendation* process is entered. That can be illustrated by:

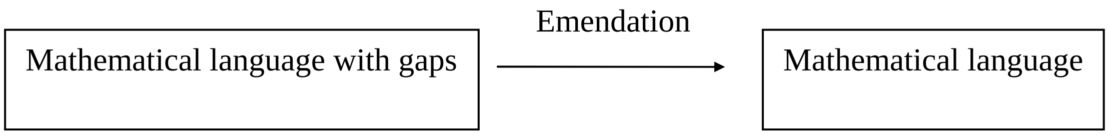


Figure 4.3 Emendation process

It is not possible, in advance, to describe the time used and the outcome of the emendation process. This process is dependent of several factors. The main factor is the text under consideration another factor is the nature of the gaps. There are certainly coherence between the text and the gaps, but it is by no mains certain that a decoding process is unambiguous with respect to different persons.

During the emendation process it is important that the focus on the text is not to narrow. It is necessary to focus on that particular gap and the corresponding part of the text, but at the same time a holistic view on the text could be of great importance. Text written elsewhere in the document could be the key to a successful emendation process. There is also the possibility that the outcome of the emendation process is 'empty'; i.e. that it is found very difficult or impossible to fill the gaps.

During the decoding or/and the emendation processes there is a risk for practising wishful thinking, to over-interpret the pupils' solutions; i.e. to 'discover or expose' more mathematics than actually embodied in the solutions. In order to eliminate or reduce the eventual over-interpretation, it is necessary 'to look back'. This means that after finishing the decoding process and the possible emendation process it is necessary to control the results against the original text. If the outcome of this control is unsatisfactory, the processes, decoding and emendation, must continue.

All of the preceding figures have a linear form, but that does not imply that the analysing process is linear. A triangular illustration of the complete analysing process seems to be more adequate.

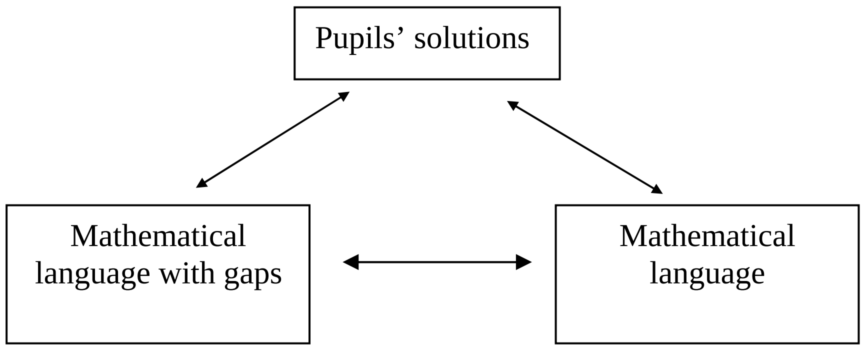


Figure 4.4 Complete decoding process

Compared with the earlier figures this one has double arrows, indicating that a 'looking back' process, a control, does not only take place at the end of the decoding process or the emendation process. The looking back processes are intertwined with both the decoding process and the emendation process it is a continuous process.

4.4 Naturalistic inquiry

Lester (1985) in the article *Methodological considerations in research on mathematical problem-solving instructions*, draws attention on four trends being of especially interest in the research on problem-solving instruction. The first one of these trends is "Shift away from experimental and toward naturalistic methodologies." (p. 52). He defines or describes a *naturalistic* inquiry by:

> (…) a research paradigm is more or less naturalistic depending on the extent to which constraints are imposed on independent, mediating, and dependent variables; (…) An "ideal" naturalistic study would impose a very low degree of constraints on these conditions and outcomes, whereas an "ideal" experimental (scientific) study would impose a high degree of constraints on them. (Lester, 1985, pp. 52-53)

and he continues:

> (…) the typical educational researcher characterizes naturalistic inquiry as follows:
> - It relies on qualitative methods.
> - It eschews rigor for the sake of relevance.
> - It relies on tacit knowledge in the formulation of theory.
> - It adopts an expansionist (as contrasted with a reductionist) stance toward research.
> - Its purpose is to discover theory and ground it in real-world data rather than verify theory. (Lester, 1985, p. 53)

From Lester & Kerr (1979) and the quotation above it follows that a study could be 'more or less' naturalistic. The grading according to the 'more or less' scale depends on the extent on which constraints are imposed on variables or the environment. An ‘ideal’ naturalistic study would impose a very low degree of constraints on the variables, while a study with complete control of the variables or the environment is called *laboratory* study. One can think of a continuum of the constraints imposed on the variables from no control to complete control (Lester & Kerr, 1979).

4.4.1 Characterisation of this study

Looking at this study under the light of the preceding characterisation of naturalistic inquiry, what is the conclusion? And, if possible, where on the ‘more or less’ scale is this study located?

Reliance upon qualitative methods

In the 1960’s critics of using quantitative methods in educational research was raised (Snow, 1974), and qualitative research methods was suggested as an alternative. At first the status of qualitative research, in education, was “marginal, and often pariah stepchild to a respected member of the research community” (LeCompte, Millroy & Preissle,1992, p.xvi). Since the late 1970’s there have been a growing critics among researches in mathematics education on the use of quantitative methods (Freudenthal, 1991; Lester, 1985; Lester & Kerr, 1979), which also Niss (2000) indirectly touched upon in his plenary address at ICME-9. He pointed to the fact that the number of studies in mathematics education based on qualitative research methods the last two decades had increased considerably, and was “today’s predominant research paradigm in mathematics education.” (Niss, 2000, p. 5).

The data collection and the analysing method, explained in sections 4.2 and 4.3, confirm that this particular study adopted a qualitative research methodology, and as such smoothly conforms to today’s predominant research paradigm in mathematics education.

Relevance versus rigour

There are, as explained in section 1.4, two main aspects of this study, firstly to uncover mathematical structures inherent in pupils’ solution procedures of some open-ended tasks, and secondly to examine if mathematical archaeology is a suitable tool for increasing our knowledge about pupils’ mathematical activity and their mathematical thinking. In the field of mathematics teaching these two aspects are relevant.

The term rigour has many facets, objectivity, internal and external validity, and reliability, on which the quality of a study relies. The objectivity and reliability concerns the data, the way the data was collected, and the findings of the study while internal and external validity to a large extent are related to the findings of the study (Miles & Huberman, 1994).

Objectivity

How data was collected is described in detail in section 4.2, section 4.2.2 and appendix A. It is explained that with one or possible two exceptions the members of the data corpus, the solutions, are the pupils' own original work. The trustworthiness or the validity to the data is therefore high. Personal assumptions, values, and biases have been addressed in section 3.2, the data are available for reanalysis, and no data has been hidden. The procedures and methods used in the study are described in sections 4.2 and 4.3. Based on this information it can be concluded that the objectivity or the degree of objectivity to this particular study is high.

Reliability

The main issue about reliability is, according to Miles & Huberman (1994), "(…) whether the process of the study is consistent, reasonably stable over time and across researchers and methods." (p. 278), while Merriam (1998) refers to reliability as "to the extent to which research findings can be replicated." (p. 205). Related to the target group, the participating classes, the researcher in this study played, as described in section 4.2.2, the role of an outsider. The researcher was not present in the classrooms when the pupils worked out the documents; he was unknown for the target group. The pupils were working on the tasks in their daily school setting. The researcher has therefore not influenced the pupils' solutions, which enhances reliability.

The design of this study ensures that reliability is high; the collection of data is explained in details in section 4.2, and in section 4.3 a detailed account for the data handling, the interpretation or the decoding process, has been given.

Another moment that strengthens reliability is the possibility to do a replicate study. The tasks solved by the pupils are available and the schools are still organised in classes, so it ought to be an easy task to organise a replicate study. However, since the target group is pupils, which is not a static group this does not mean that all the findings will be replicated. On the other hand this study includes relatively many classes, 23, which enhances the possibility of the replicability of the findings, and thus also reliability.

Internal validity

Internal validity deals with the question of credibility or trustworthiness to the findings or results of the study. According to Miles & Huberman this can be addressed as:

> Do the findings of the study make sense? Are they credible to the people we study and to our readers? Do we have an authentic portrait of what we are looking at? (Miles & Huberman, 1994, p. 278)

Merriam (1998) emphasise that internal validity deals with the relation between the findings and reality; does the findings mirror the reality? As emphasised in section 4.2 the collection of the data was done before it was decided to do an academic work based on these data. As stressed elsewhere the validity to the collected data is very high. The findings based on the interpreting of these data, and described in a mathematical language, make sense, and they are credible for the group of pupils involved in the solution activities. In some instances the interpretation of the data is not ‘straight ahead’, uncertainty sticks to the interpretation, this has been discussed in section 4.3, and it will be addressed further in the analysing chapters. The internal validity is enhanced by the fact that the findings is not located to ‘isolated isles’ in the data corpus, triangulation between the different solutions reveal that the findings are replicated across the data.

External validity

External validity deals with the generalisations of the results of a study, which is expressed by Merriam (1998) as “the extent to which the findings of a study can be applied to other situations.” (p. 207).

Most of the participants in this study were pupils in Norwegian compulsory schools; the only exception was a class from Sweden, see section 4.2.4. Compared with the number of classes in Norwegian schools at the time the data was collected, the number of classes which participated in the competition was very small, and thus also the ratio between the number of participating pupils and the number of those who did not participate. The classes, however, were as explained in section 4.2.4, located all over the country, and the available information about the level of learning to the classes indicates that the participating classes were considered as average by their teachers. This give strong indications that the findings should not be restricted to the context of the participating classes, but can be transferred to a broader context, the Norwegian compulsory schools, which is further supported by the replication of the findings across the data corpus.

As mentioned above, one of the participating classes was from Sweden, what then about the transferability of the findings to the Swedish context? The school system, the organising of the schools, in Norway and Sweden are very homogeneous, as is the mathematical syllabus in these two countries, which made it likely that the results of the study also can be transferred to the Swedish setting.

Tacit knowledge as a basis

At the time when this researcher constructed the first competition tasks, he had a relatively long practice as a mathematics teacher, all together 18 years, of these 17 years as a teacher in a college of education. As described in the preliminary chapter his teaching experience had indicated that it could be appropriate to use open-ended tasks (later classified as investigational tasks) in the competition. This was a knowledge which at that time not had been crystallized in written text; it was just an intuitive ‘feeling’. The knowledge was therefore mainly based on what Lester (1985, p. 54) characterise as “Intuitions, hunches, apprehensions and other types of knowledge”. During the teaching practice it was experienced what type of tasks which motivated and engaged the college students. It was also noticed that the students could do, what may be called, ‘a meaningful piece of mathematical work’ if they were engaged in such tasks. That was even the case for students who regarded them selves as ‘a loser in mathematics’. Even though the pupils were younger than the students at the college, it indicated that the same type of tasks could be used in the competition.

Expansionist stance

Lester characterised the expansionist stance by:

> (…) naturalistic inquirers tend to view phenomena holistically and try to avoid placing constraints on variables whenever possible. Instead of restricting their focus they believe that for complex phenomena, the whole may be different from the sum of its parts. (Lester, 1985, p. 54)

The contrast is the reductionist stance where “Scientific researchers tend to reduce their inquiry to a relatively small focus by imposing constraints on the variables involved (…)” (Lester, 1985, p. 54). The aim of this study is not to verify or reject any pre-formulated hypothesis or pre-formulated research questions, as emphasised in chapter 1 the aim is primarily to un-earth the mathematics in the already existing mathematical texts, and secondly to get an evaluation of mathematical archaeology as a tool for increasing the knowledge about pupils’ mathematical activity and their mathematical thinking. Compared with the ordinary school setting only one restriction was introduced; the activity to teachers, see section 4.2.2. Except for that restriction the pupils worked out their solutions to the tasks in their ordinary class context, which implies that one tried to avoid placing constraints on the pupils. Even though the intention was to the largest possible extent eliminate the influence of the teacher, the teacher was, as emphasised in section 4.2.2, not a controlled variable.

Discover theory

As mentioned above this study had not its origin in any pre-formulated hypothesis or pre-formulated questions, which should be verified or rejected. One objective was to uncover mathematical structures used by the pupils in their solutions procedures on some investigational tasks. Hopefully this would gain the knowledge about the pupils mathematical thinking, and thus give a contribution to the, all the time, ongoing formation or construction of the theoretically background for mathematics education.

Returning to the question put forward in the beginning of this section, the characterisation of the study in relation to the naturalistic inquiry paradigm. To summarise, the discussion has established the naturalistic character of this study. One of the variables was the teachers. Compared with an ordinary teaching situation, the activity of the teachers was restricted; i.e.

the environment to the pupils was to a certain degree controlled, see section 4.2.2. This means that the study is not an 'ideal' naturalistic study. It is however impossible to state exactly where on the 'more or less' scale of naturalistic studies this study is located, but the discussion confirms that it has to be located towards the naturalistic end of the scale.

4.5 Conclusion research methodology

The above discussion was focused against three areas: Data collection, the analysis method and the characterisation of the study. Included in the last area are the research approach and the issues of reliability and validity.

All together 25 classes responded to the tasks used in the *Tangenten* competitions, and of which 23 responses distributed on six different tasks will be analysed in this study. Two of the tasks account for more than 50% of the analysed solutions.

The competition rules instructed the teachers to play a passive role, but they did not 'eliminate the teachers'. The teachers were permitted to interpret the task-text for the pupils, but the teachers should not pose hints or questions. The competition was for the pupils not for the teachers. On the other hand the teachers were still a vital part of the classroom environment. The teachers have obviously influenced the solutions, but the degree of their influence is impossible to determine. When the pupils were working on the tasks no unfamiliar person was present, and even though their teacher had to play a passive role, the situation had not the character of a teaching experiment the pupils were all the time in their normal class situation. It is argued that the study is within the naturalistic inquiry paradigm.

The analysis process is one in which a knowledgeable person analyses the pupils' texts in terms of mathematics. If this process leads with some certainty to a known formal mathematical product then this product is declared the most probable one which conforms to the pupil-generated product.

Regarding the methodological demands of research, the trustworthiness of the analysis lies in descriptions which are so truthful and convincing so that others can in some way re-walk the analysis journey. Thus both the activity and the decoding process should be open for scrutiny by others.

The next six chapters will apply the analysis method on the pupils' mathematical texts.

5. A NUMBER AND ITS REVERSE

The first task in the *Tangenten* competition was presented in the first issue of the journal, **1**(1). It was called *A number and its reverse*. Two classes responded on this task:

Class A: A grade 5 class;

Class B: A grade 6 class.

5.1 The task

The following algorithm is given:

Choose a natural number, e.g. 18.

Write the reverse number (write the digits in opposite order), in our example 81.

Subtract the smaller number from the larger, and write the answer.

$81 - 18 = 63$.

a) Pick out some other natural numbers with two digits and carry through the algorithm. Study your answers, and look for a relationship(s) (a rule/a pattern). Write down your observations.

b) Choose a three-digit natural number and carry through the algorithm.
What are you observing?
Pick out some other numbers with three digits and do the same.
Compare with the relationship (the rule/pattern) from a).

c) Can you explain (prove) that the relationship (the rule/pattern) from a) is true?

The task can be extended in several directions.

Try and see what you can do. Try also to make new tasks.

Example

Carry out the same algorithm in an arbitrary number system, e.g. with base six.

Then the rule for base ten numbers can be compared with the rule for base six numbers.

What are you observing? Why does this happen?

Another direction of the extension is to expand the algorithm with two more steps; e.g.:

Reverse the answer of the subtraction and add this number to the answer of the subtraction.

In our example

Answer subtraction:	63
+ reverse number:	36
Answer:	99

Do the same with two-, three-, four-, ... digit numbers and study your answers. What are your findings?

5.2 Solution

In addition to the three points a), b) and c) the first suggested extension of the task, calculating in another number base than ten, will be handled since one of the classes, Class B, investigated this question. The second indicated extension of the task, reversing the answer of the subtraction and adding it to the subtraction answer, will not be discussed in this study since no one of the classes in their solutions addressed this question. This problem is however discussed very thoroughly by Gardiner (1987). In the rest of this chapter the term *number* means a natural number inclusive zero; i.e. an element in the set $\mathbb{N}_0$.

None of the two classes reported that they carried through the algorithm using *palindrome numbers*; i.e. a number read the same both backwards and forwards, for example 1221, 12321 etc. The reason is probably that they have found it of little interest to do a subtraction using two identical numbers. The answer will always be 0.

The given algorithm leads to a subtraction where it is necessary to ‘borrow’ or ‘carry’ at least once. The number of ‘borrowings’ depends both of the number of digits (i.e. the size of the number) and their mutual relationships. Since the focus of this study not is pupils’ possible difficulties with these subtractions, pupils’ possible difficulties with subtraction will not be commented on. This topic is covered by many publications; e.g. (Breiteig & Venheim, 1993; Brown, 1981; Dickson, Brown & Gibson, 1984; NCTM, 1978).

The algorithm is unique in the sense that each time it is applied on a number the result or the answer is unique, which means that the algorithm can be classified as a function. In the following the symbol S will be used for this function when it is calculated in the decimal system. Is the calculations carried out in a number system with base *g*, the symbol S_g is used. If the independent variable is called *t*, then the symbols $S(t)$ or $S_g(t)$ will represent the dependant variable.

A function can be given or represented in at least three different ways; symbolical, graphical, and numerical. The symbolical representation is usually by a formula (sometimes made up of several formulas), the graphical by a graph, and the numerical by a table. In this case a function-table means an overview between the values of *t* with the corresponding values of $S(t)$.

5.2.1 Numbers with two digits

Calculating in the decimal system

Let $t = ab_{\text{ten}}$, the domain of S, D_S (the *t*-values), consists of the two-digit numbers where the tens-digit in *t* is greater than the units-digit in *t*. The terms *units-digit*, *tens-digit* etc., means the digit at the unit position to the number, the digit at the tens position etc.. This implies that D_S consists of the following 45 numbers:

$$D_S = \{t \in \mathbb{N} \mid a - b > 0\} = \{10, 20, 21, 30, 31, 32, 40, 41, 42, 43, 50, 51, \ldots, 98\}$$

Illustrated in the 10×10-square below, the set D_S consists of all the numbers above the zigzag line in bold.

90	91	92	93	94	95	96	97	98	99
80	81	82	83	84	85	86	87	88	89
70	71	72	73	74	75	76	77	78	79
60	61	62	63	64	65	66	67	68	69
50	51	52	53	54	55	56	57	58	59
40	41	42	43	44	45	46	47	48	49
30	31	32	33	34	35	36	37	38	39
20	21	22	23	24	25	26	27	28	29
10	11	12	13	14	15	16	17	18	19
0	1	2	3	4	5	6	7	8	9

Figure 5.1. Overview of the domain of S, D_S

In their solution Class A used D_S as given above, while it is very likely that Class B excluded numbers with zero as units-digit, $b = 0$, from D_S, i.e. the left column in figure 5.1was excluded. This assumption reduces the number of elements in D_S to 36. For the further analysis it is of no importance if the number of elements in D_S is 45 or 36. Carrying through the algorithm with the number *t* gives:

$$S(t) = ab_{\text{ten}} - ba_{\text{ten}} = (10a + b) - (10b + a) = 9a - 9b = 9(a - b) \qquad (5.1)$$

Let $x = a - b$, the difference between the tens- and units-digits in *t*. It follows that $1 \le x \le 9$, and that the number of different *x*-values is nine. Consequently, the corresponding number of different values of $S(t)$ is also nine. If the value $b = 0$ is excluded, as Class B did, there are eight different values of both *x* and $S(t)$. From the preceding it follows also that the range of *S*, $V_S = S(D_S)$, is the set $M_9 = \{9,18,27,\ldots,72,81\}$ (9-table to 9·9), if the value $b = 0$ is excluded the number 81 has to be removed from the set M_9. This implies that: $S(t)$ is always a number in the 9-table, a multiple of 9.

The expression (5.1) can be formulated as: The answer, $S(t)$, is always 9 times the difference between the tens- and units-digits in *t*; i.e. numbers with identical *x*-value have identical $S(t)$-value, and numbers with different *x*-values have different $S(t)$-values. Using mathematical symbols, the above can be described by: By means of the identity $x = a - b$ the set D_S is separated into the following disjoint subsets:

$$
\begin{aligned}
A_1 &= \{t \in D_S \mid a-b=1\} = \{10, 21, 32, 43, \ldots, 98\} \\
A_2 &= \{t \in D_S \mid a-b=2\} = \{20, 31, 42, \ldots, 97\} \\
A_3 &= \{t \in D_S \mid a-b=3\} = \{30, 41, \ldots, 96\} \\
&\vdots \\
A_9 &= \{t \in D_S \mid a-b=9\} = \{90\} \\
D_S &= \bigcup_{i=1}^{9} A_i
\end{aligned}
\tag{5.2}
$$

Let $i = 1,2,3,\ldots,9$ and $j = 1,2,3,\ldots,9$. If $t_1, t_2 \in A_i$ then $S(t_1) = S(t_2)$, and if $t_i \in A_i, t_j \in A_j$ and $i \neq j$ then $S(t_i) \neq S(t_j)$. In a mathematical language this means that $x = a - b$ determine a relation, ***R***, which separates the set D_S into mutually disjoints non-empty subsets $A_i, i = 1,2,3,\ldots,9$ given by (5.2). Let $t_1, t_2 \in D_S$ and $t_1 = a_1 b_1$, $t_2 = a_2 b_2$ then the relation ***R*** is given by:

$$t_1 \boldsymbol{R} t_2 \Leftrightarrow a_1 - b_1 = a_2 - b_2 \tag{5.3}$$

It is easy to verify that ***R*** determines a decomposition of D_S into subsets $A_i, i = 1,2,3,\ldots,9$ and that ***R*** is reflexive, symmetric, and transitive; i.e. ***R*** is an *equivalence relation* on D_S where $A_i, i = 1,2,3,\ldots,9$ is the *equivalence classes*. The equivalence classes are here represented as residues modulo 11, except for the remainders 0 and 1. It is however noticed that the palindrome two-digit numbers can be included in D_S, these numbers will then constitute a new equivalence class on D_S; i.e. the remainder 0 modulo 11.

Rearranging the 10×10-square, figure 5.1 page 75, to a 11×9-rectangle, has as a consequence that each of the subsets A_i, the equivalence classes, is located to separate columns. Each subset consists of the numbers above the bold line. The table becomes then:

99		A_9	A_8	A_7	A_6	A_5	A_4	A_3	A_2	A_1
88	89	90	91	92	93	94	95	96	97	98
77	78	79	80	81	82	83	84	85	86	87
66	67	68	69	70	71	72	73	74	75	76
55	56	57	58	59	60	61	62	63	64	65
44	45	46	47	48	49	50	51	52	53	54
33	34	35	36	37	38	39	40	41	42	43
22	23	24	25	26	27	28	29	30	31	32
11	12	13	14	15	16	17	18	19	20	21
0	1	2	3	4	5	6	7	8	9	10

Figure 5.2. The equivalence classes determined by ***R***

Calculating in an arbitrary number system

Let *g* be the base in this number system, and set $t = ab_g = ag + b$. The algorithm with this *t*-value gives, analogous to (5.1):

$$S_g(t) = ab_g - ba_g = (ag + b) - (bg + a) = a(g-1) - b(g-1) = (g-1)(a-b) \qquad (5.4)$$

The expression (5.4) shows that in the *g*-number system the answer, $S_g(t)$, always is a number in $(g-1)$-table, a multiple of $(g-1)$. The result (5.1) at page 75, which stated that $S(t)$ is a multiple of 9, is then a special case of the result in the *g*-number system.
The formula (5.4) tells also that $S_g(t)$ is $(g-1)$ multiplied by the difference between the digits of *g* (the difference between the digit at the *g*-position and the units-digit). The corresponding result in the decimal system was that $S(t)$ is 9 multiplied by the difference between the tens- and units-digits in *t*.

The identity $x = a - b$ separates also in this case the domain of S_g into equivalence classes. The equivalence classes are now the residues modulo $(g+1)$, except for the remainders 0 and 1. The number of equivalence classes is therefore $(g-1)$. As was the case in the decimal system, palindrome two-digit numbers in the g-number system can be included in the domain of S_g, and quite parallel, these numbers will constitute the equivalence class with remainder 0 modulo $(g+1)$.

5.2.2 Numbers with three digits

Set $t = abc_{\text{ten}}$ where *a*, *b* and *c* are the digits in *t*. Let $a > c$, this is, as for two digit numbers no restriction. The domain of *S*, D_S, is now the set

$$D_S = \{t \in \mathbb{N} \mid a - c > 0\} = \{100, 110, 120, 130, \ldots, 200, 201, 210, 211, \ldots, 998\}$$

The set D_S contains 450 elements; i.e. there are 450 different values of *t*, if numbers where $c=0$ is excluded, as Class B did, the number of different *t*-values are reduced to 360. The algorithm gives in this case:

$$S(t) = abc_{\text{ten}} - cba_{\text{ten}} = (100a+10b+c)-(100c+10b+a) = 99a-99c = 99(a-c) \qquad (5.5)$$

The parallel between expression (5.1) and expression (5.5) is striking. In the two-digit case, (5.1), the answer is 9 times the difference between the tens-digit and the units-digit in *t*, and therefore a number in the 9-table. In the three-digit case, (5.5), the answer is 99 times the difference between the hundreds digit and the units-digit in *t*, and consequently in the 99-table, and as therefore divisible by both 9 and 11.

Let $x = a-c$, the difference between the hundreds- and units- digits in *t*. From (5.5) it follows that numbers with identical *x*-value has identical $S(t)$-value, and numbers with different *x*-values have different $S(t)$-values. As was the case for two digits numbers *x* can only take nine different values, correspondingly the number of different $S(t)$-values is also nine. This means that also in the case of three digits numbers the domain of *S*, D_S, is separated into nine equivalence classes. These equivalence classes are:

$$\begin{aligned} A_1 &= \{t \in D_S \mid a-c=1\} \\ A_2 &= \{t \in D_S \mid a-c=2\} \\ A_3 &= \{t \in D_S \mid a-c=3\} \\ &\vdots \\ A_9 &= \{t \in D_S \mid a-c=9\} \\ D_S &= \bigcup_{i=1}^{9} A_i \end{aligned}$$

However (5.5) gives no direct information about the digits in $S(t)$. For the case of two-digit numbers the answer, $S(t)$, was always a number in the 9-table. As will be known the 9-table consists of the numbers 9, 18, 27,..., 90. The pattern to these numbers can be described: For each step upward, the tens-digit increase by one, and the units-digit decrease by one. The sum of the digits is always 9. For many pupils in the 5th and 6th grades this is a well-known pattern, which implies that these numbers are familiar for the pupils. The numbers in the 99-table, the multiples of 99, have not that familiarity. In order to get closer information it is of interest to calculate the digits in $S(t)$. Another reason for calculating these digits, is that both classes, Class A and Class B, gave statements related to the digits in $S(t)$.

Remembering that $x = a-c$, calculation of the digits in $S(t)$ give:

$$\begin{aligned} S(t) = abc_{\text{ten}} - cba_{\text{ten}} &= 100a+10b+c-(100c+10b+a) \\ &= 100(a-c)+(c-a) = 100(a-1-c)+90+(10+c-a) \qquad (5.6) \\ &= (a-1-c \quad 9 \quad 10+c-a)_{\text{ten}} = (x-1 \quad 9 \quad 10-x)_{\text{ten}} \end{aligned}$$

From (5.6), and since $0 \le x-1 < 9$ and $0 < 10-x \le 9$, it follows that the hundreds digit in $S(t)$ is $x-1$, the tens-digit is 9, and the units-digit is $10-x$.

The following identity is valid:

$$9x = 10(x-1) + (10-x) = (x-1 \quad 10-x)_{\text{ten}} \tag{5.7}$$

Combining (5.6) and (5.7) gives that the two-digit number composed of the hundreds digit and units-digit in $S(t)$, $(x-1 \quad 10-x)_{\text{ten}}$, is 9 times *x*; i.e. 9 times the difference between the corresponding digits in *t*. In addition the tens-digit in $S(t)$ is 9. A complete description of the digits in $S(t)$, by means of the digits in *t*, has then been established.

Another consequence of (5.6) is that the sum of the hundreds digit and the units-digit in $S(t)$ always is 9.

5.3 Class A

5.3.1 Numbers with two digits

To the first question Class A responded:

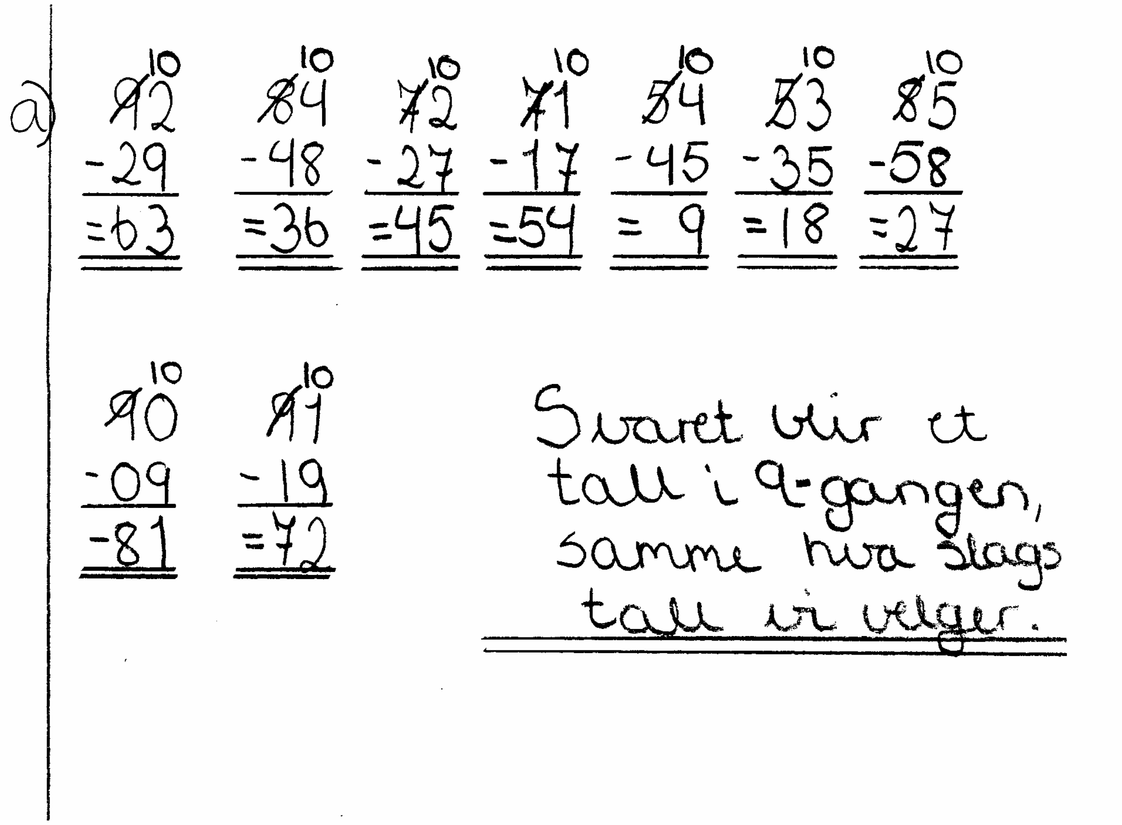


Figure 5.3. Facsimile of Class A's answer to the first question (p.1)

The Norwegian text reads:

> The answer is always a number in the 9-table, whatever number we choose.

This confirms that Class A discovered the role played by the numbers in the 9-table in this task; the answer, $S(t)$, is a multiple of 9; i.e. that V_S is a subset of M_9.

Is it possible to un-earth on which basis Class A has drawn their conclusion? Two different strategies or perspectives could have been the starting-point. One is to carry through the algorithm on all the 45 numbers in D_S and investigate the answers $S(t)$. This strategy could be called *systematic investigations*. The other strategy is to do an algebraic calculation or proof, like what is done in (5.1), this will be called *algebraic calculation*. It is noticed, at this point in their answer Class A did not give an explicit rule for the value of $S(t)$. However, the table, see facsimile page 79, is a complete overview over all the nine possible answers, $S(t)$-values, which are attainable as results when the algorithm is applied on two-digit numbers. Class A demonstrates therefore, in a way, indirectly the rule by the table presented; the answer $S(t)$ is always 9 times the difference between the tens-digit and units-digit in *t*. It is reasonable to suppose that the conclusion to Class A rests on more than the nine examples written in their solution. One reason for this assumption is that their written solution is the end product of their work on the competition task, and as such can be regarded as a summary or an overview of what the class has discovered. Another reason is that these nine examples give all the possible $S(t)$ values when the algorithm is executed on two-digit numbers, *t*. The probability that a randomly selected sample of nine numbers, *t*, from the set D_S should give this outcome is very low, ≈0,0004. (Assumption: Uniform model of probability. The calculation of the probability is then given by $\left(\prod_{i=1}^{9}\binom{i}{1}\right) \div \binom{45}{9} \approx 0,0004$.) It is therefore most unlikely to suppose that Class A only had carried thorough the algorithm with these nine numbers before their conclusion was written. On the other hand they have not stated that they carried through the algorithm on all the 45 numbers in D_S. The question of how many numbers the algorithm was executed on is, however, of little importance. What is interesting and important is that even though Class A not explicitly wrote the statement that the answer is always 9 times the difference between the tens- and units-digits in *t*, it seems reasonable that Class A has realised this fact. It is very unlikely that the table, found in their answer, a table that contains all the nine possible differences, is written by chance. The probability that a randomly selected sample of nine numbers from the set D_S should give this outcome is, as ascertained above, very low (≈0,0004), and can be neglected. In a mathematical context this means: Class A has observed that for a number *t*, where *a* is the tens-digit and *b* the units-digit, the expression $(x =)a-b$ plays a decisive role in determining the number $S(t)$. Numbers with identical *x*-value have the identical $S(t)$-value, and numbers with different *x*-values have different $S(t)$-values. As pointed out in section 5.2.1 this means that D_S is decomposed into mutually disjoint non-empty subsets, and where each subset consists of mutually equivalent elements, and elements of different subsets being non-equivalent. In a mathematical language this means that Class A has used that the relation ***R***, given by (5.3) defines an equivalence relation on D_S.

One important property of equivalence classes is that all elements in one equivalence class have the same property with respect to the given relation. Consequently, in order to establish the statement that the answer is nine times the difference between the tens-digit and units-digit in *t* it is sufficient to carry through the algorithm with one element from each of the nine equivalence classes. From the point of view, a probability approximately 0,0004 refereed earlier, it is reasonable to conclude that Class A had realised the statement that $S(t)$ is 9 times the difference between the tens-digit and units-digit in *t*, when they made their table (See facsimile page 79). Further support for this conclusion is the answer Class A has given to point c) in this task. At point c) the problem was to prove or explain the relationship stated for two-digit numbers. The objective to Class A was therefore to argue why their statement (p. 1) "The answer is always a number in the 9-table whatever number we choose." was correct. Class A's argument for the correctness of their statement was:

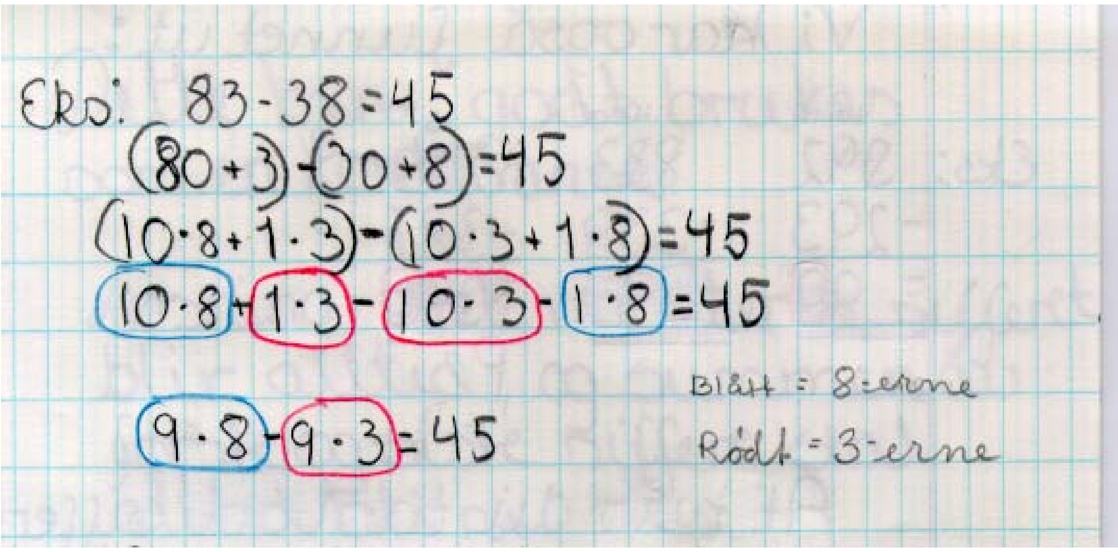


Figure 5.4. The 'proof' to Class A (p. 5)

The Norwegian text reads:

Blue = the 8's
Red = the 3's

And they continue

> 9 is factor in both multiplications. First stands 8 on the tens position, then we subtract it from the units position. Then we have 8 nine times.
> We can combine like this: $9(8-3)$
> Multiplication and division belongs together. Because 9 is a factor in the answer, the answer can be divided by 9. (Class A, p. 5)

A mathematical description of the procedure described by the class is that firstly they constructed an expression where the number *t* and the reversed number to *t*, is written on an expanded form, secondly this expression has been handled algebraically, thirdly and lastly they concluded. As seen from the quotation, parts of this algebraic activity have been given grounds for. A rough sketch of the algebraic activity carried through by the class, reveals:

- a decomposition of parenthesis with both positive and negative signs;
- organising the numbers in a suitable way;
- factorising using distributive properties.

It is noticed that Class A in the above procedure has used important properties to a ring.

Even though Class A demonstrated the statement using the number $t = 83$, their arguments have a general form. In this case it denotes that the number 83 and the digits 8 and 3 are handled as general symbols; i.e. they could be replaced by the two-digit number $t = ab_{\text{ten}}$ and the same arguments could be used. An example of this generality is for example found in the excerption:

> 9 is factor in both multiplications. First stands 8 on the tens position, then we subtract it from the units position. Then we have 8 nine times. (Class A, p. 5)

This is a well-written formulation on parts of the algebraic calculation found in (5.1) page 75. A similar written argument for the fact that there also has to be a *3 nine times* component in the linear combination, is not found in their answer. Why Class A not argued for this fact, could be difficult to figure out in retrospect. There are at least three different interpretations for the lack of this statement:

1. This is quite parallel with the 8 nine times, the argumentation are similar, the result is obvious, and further argumentation is not necessary;
2. Class A had problem with the signs. In this case they have to relate to the expression $+1 \cdot 3 - 10 \cdot 3$, which give a negative number as an answer. This causes difficulties, and they are not able to express this in a written form, as done in the 8 nine times case;
3. They have not observed, and therefore not problematised or questioned this. They have just observed that the two terms have different signs, and then maintained or believed that it goes like "8 times nine".

A closer look at their answer, see facsimile page 81, reveals that they did not have problems in dissolving parenthesis, even though a parenthesis with a negative sign. The change of the signs is correct. Since $+1 \cdot 3 - 10 \cdot 3 = -(10 \cdot 3 - 1 \cdot 3)$ the calculation can be understood to go 'the opposite direction'; i.e. adding a new parenthesis. The calculation in the parenthesis at the right side of the equal sign is then quite parallel or analogue with the former case, calculation of the 8's. In the pupils' answer the new parenthesis are indicated with different colours. The expressions, $10 \cdot 8$ and $1 \cdot 8$, are circumscribed by blue ovals, and the expressions $1 \cdot 3$ and $10 \cdot 3$ by red. This could indicate that the calculations should not cause the pupils particular difficulties, and also that the interpretation 1. is more likely than 2. or 3.. Without regard to which of the interpretations is the correct one, the pupils produce a linear combination where 9 is a factor in both terms. Then, using the distributive property this linear combination is factored, in order to prove that 9 is a factor.

The strategy used by the pupils in their argumentation is what at page 80 is called general calculation with numbers; i.e. the digits 8 and 3 are handled as general symbols. Class A did not calculate numerical answer to the expression; i.e. produced a number and established that this number was divisible by 9 by simply doing the division. They demonstrated that the calculated answer, $S(83)$, had 9 as a factor, and therefore can be divided by 9. This was done without doing the multiplications, additions and subtractions. Their argumentation is valid also in the general case.

5.3.2 Numbers with three digits

The agenda for point b) in this task was to observe what happens when the algorithm was executed on three-digit numbers, and then compare the observation with the relationship(s) stated in the first part of the task. The goal for the investigation was therefore in a way directed by the relationship(s) stated in the first part of this task. This relationship was “The answer is always a number in the 9-table, whatever number we choose“ (see page 79). From their answer it appears that they have adopted the goal: Investigate if this relationship also was valid for three-digit numbers. The way this investigation was implemented was that a number *t* was chosen, then $S(t)$ was calculated, and at last it was examined if 9 divided $S(t)$. In their answer are given five examples. After the first example is written “We see that the answer to this problem is divisible by 9” (Class A, p.2). After the 5th example they concluded “The relationship we found in point A) is valid also in point B)” (Class A, p. 3).

The sentence “We see that the answer to this problem is divisible by 9” indicates that Class A did not conjecture that the relationship stated for two digits number also was valid for three digit numbers. It seems rather that for Class A it was an open question if the relationship was valid, and the first result indicated that it could be valid also for three digit numbers. Their conclusion was first stated after the last given example in their answer.

As was argued in the case of two digit numbers, it is also in this case reasonable to suppose that the conclusion to Class A is based on more than the five examples given in their answer. Even though it is assumed that Class A based their conclusion on more than these five examples, it is most unlikely that they have investigated if the relationship was valid for all the 450 possible *t* values in the set D_S, see page 78.

Let $t = abc_{\text{ten}}$, where *a, b* and *c* are the digits at the hundreds-, tens-, and units-positions, $a > c$, and $x = a - c$. Section 5.1.2 stated that D_S can be separated into 9 (different) equivalence classes. Numbers with identical *x*-value form (constitute) these equivalence classes. It was also noticed, in the same section, that this result is quite analogues with the result for two-digit numbers. As argued elsewhere, section 5.3.1, it seems reasonable to suppose that they had knowledge about the equivalence classes in the case of two-digit numbers. A natural question is therefore if Class A was aware of the corresponding result for three-digit numbers?

A comparison between their answer for this part and that for the first part of the solution, reveals marked differences. In this part Class A is not demonstrating the same clarity as in the former part. This is best seen by the fact that they here have chosen five *t* values, but not all of those five *t* values are in different equivalence classes. Two of these values, 551 and 804, are in the same equivalence class. If the calculations made at the end of their answer are included, they have calculated $S(t)$ for eight different *t*-values and where the *t*-values are found in five different equivalence classes. In the former part, two digit numbers, they had exactly one representative from each of the nine equivalence classes. In this case the selection of *t* values seems therefore more arbitrary. They have not picked up members in each of the equivalence classes, and they had chosen some values in the same equivalence classes. This could indicate that Class A not realised the relationship between the digits in *t*; i.e. the difference $x = a - c$, and the hundreds and units-digits in $S(t)$, see (5.6). There is further support for this indication. At the end of their answer, page 6, Class A gave the following rule for calculating $S(t)$:

> If you take away the digit in the middle (9), it becomes always something in the 9 table, if you look at the numbers at the side. The sum of the hundreds and units-digits is always 9. (Class A, p.6)

This rule was illustrated by three examples, and from these examples it is evident that Class A here give a description of the $S(t)$-value. Even though the above rule is more detailed than the result which stated that $S(t)$ was in the 9-table, it does not 'link' the digits in *t* with the digits in $S(t)$. The above rule was followed by an activity constructed by the class.

> Write a three-digit number. Write it in the reversed order and subtract the smallest number from the largest number. Calculate the last digit. Try to find the complete answer without calculating more on the paper. (Class A, p.6)

The formulation of this activity supports the claim that Class A did not realise the relationship between the digits in *t* and $S(t)$ given by (5.6). This is primarily grounded on the fact that Class A write that the units-digit ("the last digit") in $S(t)$ shall be calculated. On that basis the remaining digits in $S(t)$ shall be determined. The proposed activity is based on relationship(s) between the digits in $S(t)$, and not between the digits in *t* and $S(t)$. If Class A had realised the relationship between the digits in *t*, and the digits in $S(t)$, an alternative or different formulation of the above activity would most likely be expected. However, the task in the competition did not ask for an explanation or a proof for the rule in the case of three-digit numbers, as it did for the two-digit case. Therefore it is quite natural that the class not gave or wrote an explanation in the three-digit case, it was not asked for. Consequently, the existing information in the three-digit case is not comparable with the information found in the two-digit case. If Class A was aware of the relationship between the digits in *t*, and the digits in $S(t)$, for three-digit numbers, seems therefore difficult to decide. This particular task had an open-ended formulation, and did not invite to look for a specific relationship between the digits in *t* and $S(t)$.

In order to establish if the $S(t)$-values was multiples of 9, as found for numbers with two digits, Class A divided each of the calculated $S(t)$-values with 9. Looking at the outcome of these divisions, they observed "(…) the answer in the divisions (control calculations) was always two identical digits. They were therefore answers in the 11-table." (Class A, p. 4). This is a correct observation. See page 78. They continued, and concluded "Earlier we know that these numbers were divisible by 9. Then the numbers had to be divisible by 9·11, consequently 99." (Class A, p. 4). This is a correct conclusion, the $S(t)$-values are multiples of 99, as stated by (5.5).

5.3.3 This is the result, what is the task?

The task had also a formulation that invited and encouraged the pupils to make or create new problems. Class A grabbed this invitation. By using their knowledge about the outcome of the algorithm used on three-digit numbers, they constructed the activity mentioned at page 84, and suggested that this task could be used as a mathematical game or a competition. This shows another dimension of school mathematics than the traditionally one. Traditionally a task or an exercise has been presented for the pupils. The activity to the pupils has been to solve those tasks or exercises. In this activity, Class A in a way suggests to move in the opposite direction. Their challenge is now to construct a task on the basis of earlier acquired knowledge. An activity like what Class A demonstrates here can be described by: This is our knowledge, in what way can it be used; i.e. what activities or tasks can be constructed?

Class A gave the following solution to the proposed task "The number in the middle becomes always 9, and the sum of the first and last digits shall always become 9." (Class A, p. 7). Dealing with the subject like that reveals that the pupils have got a close relation to the subject, and the conceptions involved. They had to be familiar with the subject matter.

5.3.4 Summary – Class A

The structure of the task guided the pupils to explore the algorithm or the outcome of the algorithm, $S(t)$ for different numbers, *t*, and search for patterns or relationships. This implied that the pupils had to relate to variables, one independent, *t*, and one dependent, $S(t)$.

Mathematically they had to handle a function in one variable. In addition they should look for relationships between the numbers *t* and $S(t)$, or between the digits in *t* and the number $S(t)$.

For two-digit numbers it is argued that Class A discovered the relationship between the digits in *t* and the number $S(t)$, described by (5.1); i.e. that $S(t) = 9(a-b)$, where *a* and *b* are respectively the tens- and units-digits in *t*. The basis for this discovery was that Class A realised that the domain of *S*, D_S, with help of $x = a - b$ could be separated into nine mutually disjoint and non empty subsets, and that all numbers *t* in one subset had identical $S(t)$-value, and that numbers in two different subsets had different $S(t)$-values. In this particular task they discovered that

$$\begin{aligned} \text{If} \quad & x = 1 \text{ then } S(t) = 9 \\ & x = 2 \text{ then } S(t) = 18 \\ & x = 3 \text{ then } S(t) = 27 \\ & \vdots \\ & x = 9 \text{ then } S(t) = 81 \end{aligned}$$

In mathematical terms they used a relation that was an equivalence relation, which is both a well known and traditionally example of mathematical structures.

It is also argued that it is reasonable to conclude that Class A demonstrated their statement through a general proof, a proof, which is analogue with what is found in (5.1). This is found to be remarkable, taken the pupils' age into account. In the autumn in 5^{th} grade, most of the pupils were 11 years old, but some would still be 10 years old. When arguing for the correctness of the their rule for two-digit numbers, Class A made use of writing numbers on expanded form, distributive law, multiplication and the links between multiplication, factor and divisibility. Mathematically, they had used the algebraic structure ring.

For numbers with three digits the picture is not as clear as it is for two-digit numbers. As known from chapter 5.2.2 the actual set of three digit numbers, D_S, can also be separated into nine equivalence classes, if Class A had realised this fact is more uncertain. Most likely they did not realise the relationship between the digits in the starting number *t*, and the resulting number $S(t)$. However they discovered a relationship between the digits in $S(t)$, and on that basis they formulated a task, and also the solution to this proposed task.

The mathematical structures inherent in the solution procedure to this task are:

- Functions;
- Equivalence relations;
- Equivalence classes;
- Algebraic structures.

5.4 Class B

In addition to answer the points a)-c) in the task, Class B continued their investigations in two different directions. One direction was to calculate in number systems with bases different from 10, this was an extension indicated in the text. The other direction was to execute the algorithm on both four- and five-digits numbers, this extension was not proposed in the text.

As emphasised in section 4.2.2 the answer to Class B was written by their teacher. A consequence of this is that the explanations in this answer are well formulated and therefore easier to understand. However, this does not mean that it is the teacher's solution under consideration. The solution has a form that may be characterised as a report, or a summary, of working procedures and the results obtained by the class. This is particular the case for the first part of the solution, when they investigated using two-digit and three-digit numbers. In this part the teacher played the role of a 'non-participating' reporter. For the last part of the solution, investigations using four-digit and five-digit numbers as input in the algorithm, the role played by the teacher is more questionable. In these cases one has to take into consideration the mutual relationships between the digits in the starting number, the number of 'borrowings'. For each type of numbers, four- and five-digits numbers, it is necessary to discuss several cases, and compared with the two- and three-digits numbers, the results are not so easy to express. In the present case these results are very well explained, the written formulations have a quality far above what could be expected by a pupil of grade 6. Under these existing circumstances it is decided not to include this part of the solution to class B in the analysis.

As noticed at page 75 it seems reasonable to suppose that Class B has omitted numbers with units-digit zero. The main argument for this supposition is a statement found in their investigation of the two-digit case. Class B stated here that $8 \cdot 9 = 72$ is the highest possible value for $S(t)$, but as shown in section 5.2.1 $t = 90$ gives $S(90) = 9 \cdot 9 = 81$. Further support for this supposition is found in the fact that in their investigation of the algorithm, for numbers with two-, three-, four- or five-digits, Class B has calculated $S(t)$ for more than 30 values of *t*, none of these *t*-values have zero as the units-digit.

In their solution Class B used terms like, 'the difference of the tens', 'the difference of the hundreds', 'the distance between the hundreds', etc., which means the difference between the tens (hundreds) in the number *t*, and the tens (hundreds) in the reversed number to *t*.

5.4.1 Numbers with two digits

Calculating in the decimal system

Class B gave this answer:

> (…) the pupils realised quickly that the result always was a number in the nine-table.
>
> On further investigations they realised that there was a relationship between the difference of the tens and the answer. Was the difference one tens the answer was $1 \cdot 9 = 9$. Was the difference two tens the answer was $2 \cdot 9 = 18$. With a difference of three tens the answer was $3 \cdot 9 = 27$ etc. It continued like that until the highest possible, $8 \cdot 9 = 72$.

Here are some examples:

21	42	51	82	91
– 12	– 24	– 15	– 28	– 19
= 9 (=1·9)	= 18 (=2·9)	= 36 (=4·9)	= 54 (=6·9)	= 72 (=8·9)

The sum of the digits is always 9. (Class B, p.2)

The above quotation can be separated into the following three statements:

1. $S(t) \in M_9$;
2. $S(t) = 9(a - b)$, where *a* is the tens-digit and *b* the units-digit in *t*;
3. The sum of the digits in $S(t)$ is 9.

The first statement is found in the sentence "... the result always was a number in the nine-table", and it shows that Class B discovered, as Class A also did, the role the numbers in the 9-table is playing in this particular task. The second statement is identical with (5.1), page 75, and is a condensed version of the relationship Class B describes with the number examples. The third statement is the last sentence in the quotation. Class B has not stated explicitly it is the sum of the digits in $S(t)$ which is 9, but the framing of the task and the context in which this sentence is found, make it most likely that it is tacitly understood that this is a statement concerning $S(t)$. In this context it is interesting to look at the solving strategy or the solution procedure to Class B.

The first part of this task directed the pupils to calculate $S(t)$ for an unspecified number of *t*-values, and look for relationship(s). The answer to Class B gives no information about the number of calculations, the number of $S(t)$-values. Since Class B omitted numbers with units-digit zero, the domain of *S*, D_S, consists of 36 elements, if Class B calculated $S(t)$ for these 36 different *t*-values or not, is of little importance for the further analysis. In their solution nine $S(t)$-values are given; the first four presents only the $S(t)$-value, while the last five gives the complete calculations. However it is stressed that these "are some examples". This indicates that Class B has calculated more than these nine $S(t)$-values. Mathematically they have produced a function table.

The second statement to Class B, $S(t) = 9(a - b)$, reveals that Class B observed that the difference $(x =)\, a - b$, where *a* and *b* are the digits in *t*, plays a decisive role in this task. They writes explicitly, see quotation page 88, that "Was the difference one tens the answer was $1 \cdot 9 = 9$. (…)". This expresses directly that numbers with identical *x*-value has identical $S(t)$-value, and indirectly that numbers with different *x*-values have different $S(t)$-values. As pointed out in section 5.2.1, and for Class A (page 80), it is very likely that Class B used the relation ***R*** given by (5.3) and separated D_S into, in this case, eight equivalence classes. Additional support for this assertion is found in the order Class B has written the calculations, see quotation page 88. In each of the five written calculations the difference, $x = a - b$, is marked as the first factor in the product, $(a - b) \cdot 9$, and the calculations are ordered according to the increase in the *x*-values, which is a natural way to organise a table. Starting with $x=1$, and ending with $x=8$, the highest difference since this class excluded numbers with units-digit zero, see page 87. As mentioned above there are all together eight different outcomes of these calculations (eight equivalence classes), and Class B has in their answer included five out of these. In addition they have stressed that these calculations "are some examples" (Class B, p. 1); i.e. not all the calculations have been exemplified. Besides this one can not leave out of account a typographical element, there was only space for five calculations on one line. From the above it can be concluded that it is very likely that Class B discovered that the relation ***R*** separated D_S into equivalence classes, and that they in their solution procedure have used this mathematical structure. The conclusion to Class B, manifested in their rule, is based on their observations. Unlike Class A, they did not argue or give a proof for the correctness of the relationship. They just commented that "(…) it had to be so because 9 is the biggest one-digit number" (Class B, p. 4).

Calculating in an arbitrary number system

One of the proposed extensions to this task was to carry through the algorithm in a number system with another base than ten. One reason for this proposal was to draw the attention to the fact that some mathematical results, theorems, are dependant on the base to the number system, or what would be more correct to say, that these theorems had to be revised going from one number system to another. One example of such a result is the well-known relationship between the divisibility by 9 and the sum of the digits to the number.

As shown in section 5.2.1 the outcome of this algorithm are dependent of the base used. The relationship (5.4) tells that if the number system has base *g* then the answer of the algorithm is given by $(g-1)(a-b)$, where *a* is the digit at the *g*-position and *b* the units-digit. The calculations found in the answer, especially the 'borrowings' or the 'carryings', shows that it is not longer a question about tens, but about sixes-, fives-, or eight's-positions, depending on the base used. They discovered that 9 no longer was the 'important' number (Class B, p. 4-5). For instance they wrote:

> 5 was the important number in the number-system with base 6
> 2 was the important number in the number-system with base 3
> etc. as 9 was the magic number in the number-system with base 10 (Class B, p. 4-5)

This is a strong indication that the 'magic' number is the number one less than the base to the number system. The answer to Class B indicates that they had realised that there exist theorems in mathematics, which are dependent of the base used for representing the numbers. This is primarily indicated in the way Class B carried through the algorithm in other number systems than a base ten system.

5.4.2 Numbers with three digits

The investigation Class B carried through in the first part of this task, the case of two-digits numbers, resulted, as mentioned at page 88, in three relationships. The next problem was to explore the algorithm applied on three-digits numbers, and also compare the relationship(s) found in this part, with the relationship(s) found in the first part. It is reasonable to suppose that the relationships found and stated for two-digits numbers would have impact on what to look for in the second part, the case of three-digits numbers.

One of the relationships stated by Class B in the first part, was a relationship between the digits in the starting number, *t*, and the digits in the ending number, $S(t)$. It is therefore quite natural that Class B in this second part directed their attention to possible relationship(s) between the digits in the starting number, *t*, and the ending number, $S(t)$.

The first part of their solution procedure was, as in the case of two-digits numbers, to carry through the algorithm on different numbers *t*. In a mathematical language they constructed a table of ordered couples, $(t, S(t))$. Secondly this table was investigated. The investigation revealed that there was a relationship between the difference $(x =)\, a - c$, the digits at the hundreds- and units-position in *t*, and the digits in $S(t)$. In fact they gave two different versions of this relationship. The first version:

> The tens become always 9.
> The difference between the hundreds multiplied by 9 gives the answer to the hundreds place and the units place. (Class B, p. 3)

The second one:

> The difference between the hundreds multiplied by the number in the middle (9) gives the outermost digits. (Class B, p. 3)

The first version of this relationship was illustrated by four different number examples. The statement "The difference ... multiplied by 9 gives the answer to the hundred place and the unit place" is declared more explicitly in the second version. Their claim is that the relationship between the hundreds-digit and units-digit in *t*, and the corresponding digits in $S(t)$ is given by $9(a-c)$; i.e. the units-digit in $S(t)$ is identical with the units-digit in $9(a-c)$, and the hundreds-digit in $S(t)$ is identical with the tens-digit in $9(a-c)$, and that the tens-digit in $S(t)$ is always nine. The relationship given by Class B is correct, and it is a complete description of the rules given by (5.6) and (5.7) (page 79).

From their mathematics lessons the pupils were well aware of the fact that multiplication is a unique operation, and since Class B excluded numbers with zero as units-digit, $c = 0$, the number of different differences, *x*-values, is eight, and thus also the number of different $S(t)$-values. Indirectly the statement to Class B says; different *x*-values gives different $S(t)$-values and identical *x*-values gives identical $S(t)$-value. As was the case for two-digits numbers Class B used that $(x =) a - c$ and separated D_S into eight equivalence classes. See section 5.2.2. Class B does not argue for the correctness of their statement out over presenting number examples which demonstrated the maintained relationship.

5.4.3 Summary – Class B

As pointed out in the summary for Class A, section 5.3.4, the structure of the task guided the pupils to investigate the given algorithm for different numbers. Consequently, the pupils had to relate to variables, one independent, *t*, and one dependent, $S(t)$, the outcome of the algorithm. In a mathematical language they handled a function in one variable.

For numbers with two digits, Class B stated explicitly that the relationship between *t* and $S(t)$ was given by $S(t) = 9(a-b)$, where *a* and *b* was respectively the tens- and units-digit in *t*. This is identical with the relationship found in (5.1). The basis for this was that Class B discovered that $(x =)a-b$ separated the domain of *S*, D_S, into disjoint and non-empty subsets, and that all members, *t*, in one subset had identical $S(t)$-value, and that numbers in different subsets had different $S(t)$-values. Mathematically D_S was separated into equivalence classes. Since Class B excluded numbers that had zero as the units-digit, they operated with 8 equivalence classes. Class B did not argue for the correctness of their statement out over saying that they had observed this relationship.

For numbers with three digits Class B gave a correct written description of the digits in $S(t)$ on the basis of the digits in *t*. This description is quite parallel with what is found in (5.6) and (5.7). As for two-digits numbers Class B has also in this case separated the domain of *D*, D_S, into 8 equivalence classes. In this case the difference between the hundreds-digit and units-digit in *t* was used to separate the numbers into equivalence classes.

Mathematical structures inherent in the solution procedure of this task are:

- Functions;
- Equivalence relations;
- Equivalence classes.

5.5 Summary – A number and its reverse

The structure of the task leads the pupils to explore the algorithm for different numbers, and search for patterns or relationships. This meant that that the pupils in their solution procedures had to relate to variables, one independent and one dependent. Mathematically they handled a function in one variable.

The classes discovered that a number could be classified by using a difference between some of the digits to the number, and that this classification separated the numbers into disjoint sets. In mathematical terms they used a relation, who was an equivalence relation, and this relation determined the decomposition into mutually disjoint non-empty subsets, equivalence classes.

In the case of two-digits numbers one of the classes, Class A, in their arguing for the correctness of their rule made use of writing numbers on expanded form, distributive law, and properties to division. Mathematically, they had used characteristic qualities to the algebraic structure ring.

An overview of the mathematical structures inherent in the solution procedures to the classes is given in the following table:

Class / Mathematical structure	A	B
Functions	X	X
Equivalence relations	X	X
Equivalence classes	X	X
Algebraic structures	X	

Table 5.1 Overview of the mathematical structures inherent in the solution procedures to the task *A number and its reverse*

6. A SET OF TEN CUBES

The second task in the competition was in issue **1**(2) of the journal. This was the competition task with the highest number of competing classes. All together seven classes, from grad 1 to grade 9, responded to this task. These classes are named from C to I, and they belongs to the grades:

Class C: A grade 1 class;
Class D: A grade 3 class;
Class E: A grade 7 class;
Class F: A grade 7 class;
Class G: A grade 7 class;
Class H: A grade 7-8 class;
Class I: A grade 9 class.

The idea to this task was found in Burton (1986).

6.1 The task

A child has a set of ten cubes.

One cube has 1 cm edge
one cube has 2 cm edge
one cube has 3 cm edge
until the largest cube with 10 cm edge.

The child tries to build two towers of equal height.
Is it possible for the child to do that construction if it wants to use all the cubes?
Show how, or explain why it can not be done.

If the largest cube, the cube with edge 10 cm, is removed, is it then possible to build two towers of equal height (still using all the remaining cubes)?

What, if also the cube with 9 cm edge is removed?

What about the building of two towers of equal height if it in addition to the ten original cubes a cube with edge 11 cm is added?

Investigate this problem for other numbers and try to find a rule/ a pattern.

Use the rule/pattern and solve this problem:

If we have a set of twenty different cubes with edges from 1 cm to 20 cm, is it possible to build two towers of equal height, using all the cubes?

Try to explain why your rule is true.

An extension of the problem could be:
If it is possible to build two towers of equal height with a set of 11 different cubes [with edges from 1 cm to 11 cm], find as many ways as possible to do this construction.

6.2 Solution

The mathematical challenge to this task is investigation of sums of all natural numbers up to a certain number, or to be more specific to investigate when two different sums of different natural numbers are equal; i.e. the calculation of each of the sums shall give the same answer. That the two sums are different means in this respect that all the elements in one sum are different from all the elements in the other sum. It turns out that a central number in this context is the sum of the natural numbers from 1 to an arbitrary number *n*. In the mathematical literature this sum is frequently denoted T_n, and it is defined by:

$$T_n = 1 + 2 + 3 + ... + n \tag{6.1}$$

As is well known the numbers T_n are called the *triangular numbers*. In the rest of this chapter the term number means a natural number; i.e. an element in the set $\mathbb{N}$.

The solving strategy used by most of the classes was to investigate the parity to T_n, and it can be shown that a necessary and sufficient condition for building two towers of equal height is that T_n is an even number, which occur if and only if $n = 4k$ or $n = 4k - 1$ where $k \in \mathbb{N}$ (Torkildsen, 1993b). The necessity of this condition is trivial, contrary to the sufficiency. All solutions had in a way the parity of T_n as a basis, but none of the solutions that explicitly used the parity of T_n as a solving strategy commented on the sufficiency. They took it for granted that the evenness of T_n also was a sufficient condition for building the two towers of equal height. This logical shortcut implied that the problem was reduced to or transformed to investigate the parity of T_n. Three different solving strategies have been identified:

1. Comparison;
2. Use of table;
3. Study of the formula for T_n.

The first of these strategies, comparison, does not explicitly focus on the parity of T_n, as is the case for the other two. Since the classes that used one of these two strategies, 2. or 3. above, tacitly supposed that the building was possible if T_n was an even number, the solutions demonstrated below, for these two strategies, concentrate only on determining the parity of T_n. The sufficiency aspect of the problem will not be argued for in this study, for a proof see for example Torkildsen (1993b).

6.2.1 Comparison

This solving strategy can be described thus:

To verify that two towers have equal height, demonstrate how to build the towers.

To verify that it is impossible to build two towers of equal height; use all the cubes and build two towers of arbitrary heights. If the difference between the two heights is an odd number it is impossible to build two towers of equal height.

This is a strategy that leads to a correct solution, as can be seen from the following. Separate randomly the numbers $1, 2, 3, ..., n$ into two different sets $S_1 = \{a_1, a_2, ..., a_r\}$ and $S_2 = \{a_{r+1}, a_{r+2}, ..., a_n\}$ where all the a_i 's, for $i = 1, 2, 3, ..., n$, are different. Let $h_1 = a_1 + a_2 + ... + a_r$ and $h_2 = a_{r+1} + a_{r+2} + ... + a_n$, then $T_n = h_1 + h_2$.

It is no restriction to suppose $h_1 \geq h_2$. The difference, *d*, between the heights of the two towers is given by $d = h_1 - h_2$ or $h_1 = h_2 + d$. This implies that $T_n = 2h_2 + d$, and consequently if *d* is an odd number, T_n is an odd number, and it is impossible to build two towers of equal height.

This method can be executed directly on the cubes, and it is therefore possible to determine the impossibility of building two towers of equal height without doing formal additions.

6.2.2 Use of table

A table of the first ten triangular numbers:

n	1	2	3	4	5	6	7	8	9	10
T_n	1	3	6	10	15	21	28	36	45	55
Parity of T_n	*odd*	*odd*	*even*	*even*	*odd*	*odd*	*even*	*even*	*odd*	*odd*

Table 6.1. The parity of the smallest triangular numbers

The pattern that appears in the bottom row in table 6.1 repeats the sequence *odd*, *odd*, *even*, *even* cyclically. If this pattern continues, T_n will be even for each 3rd and 4th number *n*; i.e. for $n = 4k + 3$ and $n = 4k$, $k \in \mathbb{N}_0$. It can be demonstrated that this is a correct result.

The triangular numbers T_n can also be given by the recursive formula:

$$T_1 = 1 \text{ and } T_n = T_{n-1} + n \quad \text{for} \quad n > 1 \tag{6.2}$$

Combining (6.2) and the fact that two consecutive natural numbers have opposite parity, gives:

$$
\begin{aligned}
T_1 &= 1 && , \ odd \\
T_2 &= T_1 + 2 && , \ odd + even = odd \\
T_3 &= T_2 + 3 && , \ odd + odd = even \\
T_4 &= T_3 + 4 && , \ even + even = even \\
T_5 &= T_4 + 5 && , \ even + odd = odd \\
T_6 &= T_5 + 6 && , \ odd + even = odd \\
&\vdots
\end{aligned}
$$

This shows that the parity expressions to T_2 and T_6 are identical, which means that the length of the repeating cycle is 4. Consequently, since T_3 and T_4 are even, T_n will be even for $n = 3 + 4k = 4k + 3$ and $n = 4k$, $k \in \mathbb{N}_0$.

It is noticed that the periodicity and recurrence imbedded in the pattern can be applied to establish the sufficiency condition for the building of the twin towers. Since adding the two next numbers always means to add an odd number the parity always changes two steps further. And more: Since four consecutive numbers $a, a+1, a+2$ and $a+3$ always can be paired so that $a + (a+3) = (a+1) + (a+2)$ the possibility to make twin towers will repeat itself with a cycle of four. Since it is impossible to make the twin towers for $n = 1$ and $n = 2$, and possible for $n = 3$ and $n = 4$, $1 + 2 = 3$ and $1 + 4 = 2 + 3$, this pattern will repeat itself with a period of four. This shows the sufficiency.

6.2.3 Study the formula for T_n

It is well known that $T_n = \dfrac{n(n+1)}{2}$.

The question is for which numbers *n* is T_n an even number?

T_n even implies that $\dfrac{n(n+1)}{2} = 2t$ for a $t \in \mathbb{N}$. This means that $n(n+1) = 4t$. Since n and $n+1$ are two consecutive natural numbers, exactly one of them is even. This implies that either $4 \mid n$ or $4 \mid (n+1)$ which is equivalent with $n = 4k$ or $n = 4k - 1$ where $k \in \mathbb{N}$.

6.2.4 Extension of the task

A proposed extension of the task was, for the case $n = 11$, to examine in how many ways it was possible to construct two towers with equal height. In this case $T_{11} = 66$, which implies that the height of each of the twin towers is 33. Mathematically this is equivalent to determine the number of sums, which satisfies

$$\sum a_i = 33 \text{ where } a_i \in \{1,2,3,\ldots,11\} \text{ , and all the } a_i\text{'s are different.}$$

It follows then that the remaining numbers less than or equal to 11 that are not used in the sum above also add up to 33. It is tacitly understood that orders does not matter; i.e. sums that contain the same components are identical; i.e. $3 + 9 + 10 + 11$ and $9 + 3 + 10 + 11$ are identical.

This problem can be made a bit simpler. Since 11 have to be located in one of the twin sums, it is sufficient to determine the number of sums that contains 11 as one of the components. Accordingly to determine the number of sums that satisfies:

$$\sum a_i = 22 \text{ where } a_i \in \{1,2,3,\dots,10\} \text{ , and all the } a_i \text{'s are different.}$$

Let k be the number of components in the sum above. It follows that $3 \le k \le 6$. The number of sums can be explored systematically, and it turns out to be 35 different sums containing 11 as one of the components.

Mathematically this is a *partition problem*, and it can be solved using *generating functions*. Problems of this type can result in very comprehensive calculations, which are not difficult, but the extent of the calculations implies that the problem is not trivial. See for example Grimaldi (1994, pp. 446-447). For this particular task the generating function, $p(x)$, is given by $p(x) = \prod_{i=1}^{10} (1 + x^i)$. The number of twin sums will then be the coefficient of x^{22}.

Generally, if T_n is an even number, the number of twin towers with height $\frac{T_n}{2}$ is given as the coefficient of $x^{\frac{T_n}{2} - n}$ in $\prod_{i=1}^{n-1} (1 + x^i)$.

6.3 Class C

Class C, a grade 1 class, worked on this task after they had been at school for 2-3 months. The pupils, or most of them, were at that time not familiar with all the letters in the alphabet. It was therefore necessary that the teacher assisted in the writing process (Høines,1991).

Since the pupils were in grade 1, and in addition it was relatively early in the school year, summations of type (6.1) for n greater than three or four would most probably be difficult to perform for a large group of the pupils in this class. In addition their knowledge about the parity for numbers above 10 would most likely be very limited. This is supported by the fact that Class C for the case $n = 3$, used that $T_3 = 6$ can be divided in two equal parts, but they did not do that for larger numbers. However, this class worked out a solution without performing additions of type (6.1) and without mentioning the terms odd or even numbers. Their solution procedure was based on a combination and manipulating of concrete material, *Cuisenaire* rods, and drawings on large sheets of paper. In this manner they decided if it was possible or not to build two towers of equal height. These results were put in a table:

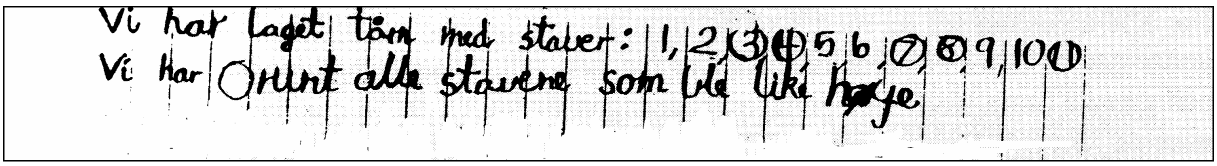

Vi har laget tårn med staver: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
Vi har ordnet alle stavene som ble like høye

Figure 6.1. The table constructed by Class C (*Tangenten* **2**(2), p. 5)

The Norwegian text reads:

> We have made towers with rods:
> 1,2,3,4,5,6,7,8,9,10,11
> We have O around all the rods that were of equal height

This use of table is in many respects analogue with the use of table in the former task, *A number and its reverse*. Class C operates here with:

- an independent variable, n, the number of rods;
- a rule, f, the building of two towers of equal height;
- a dependant variable, $f(n)$, with values *yes* or *no* (possible/not possible to build the two equal towers).

Since this rule is unique it is a function in one variable, and can be written as

$$f(n)=\begin{cases} \textit{yes} \text{ if it is possible to bulid the twin towers} \\ \textit{no} \text{ if it is not possible to bulid the twin towers} \end{cases} \tag{6.3}$$

If $f(n)=yes$ Class C circumscribed the independent variable n, and if $f(n)=no$, n was not circumscribed. In this table, figure 6.1, they discovered a pattern. This pattern was then applied to decide if it was possible to make two towers of equal height, using twenty different cubes. Expressed in their words:

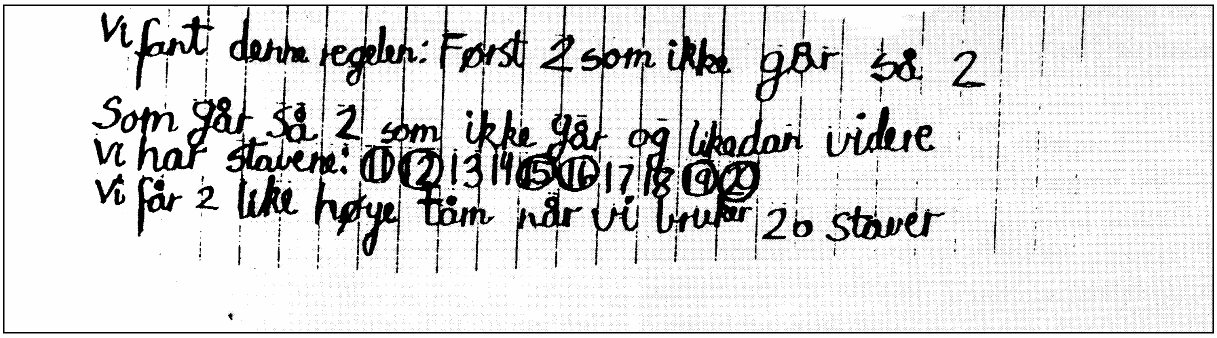


Figure 6.2. Stating and applying the pattern, Class C (*Tangenten* **2**(2), p. 5)

The Norwegian text reads:

> We found this rule: First 2 that not goes then 2
> that goes then 2 that not goes and it proceeds just the same
> We have the rods: 11,12,13,14,15,16,17,18,19,20
> We get 2 towers of equal height when we use 20 rods

In the quotation above, is found the sentence ‘and it proceeds just the same’. It should be noticed that a direct translation into English of the Norwegian sentence used by Class C is problematic. In means that something continues in the same manner, but the corresponding Norwegian word to *continue* was not used. The natural interpretation of this is: The rule discovered and described for the first 11 numbers, is valid for the next numbers without any reservation or limitation. The reasonableness of this interpretation is further emphasised that Class C applied this rule in determining the value to $f(20)$. Using mathematical symbols the rule stated by Class C can be written:

$$\begin{aligned} &f(1) = f(2) = no \\ &f(3) = f(4) = yes \\ &f(n) = f(n-4) \text{ for } n \geq 5 \end{aligned} \tag{6.4}$$

In mathematical terms this is a recursive-defined function.

The solving procedure to Class C rests on two solving strategies, the use of a table and comparison. The main solving strategy was the use of a table. Comparison was applied in their argumentation for the correctness of the $f(n)$-values for the cases $1 \leq n \leq 11$; i.e. when they completed their table. After the table had been worked out, it was explored, and based on this exploration a conjecture, (6.4), was put down. This conjecture was then used to conclude that $f(20) = yes$, which is correct.

As referred above, Class C used rods and drawings to calculate $f(n)$. It is, however, a main difference in using rods and drawings to prove that $f(n) = yes$; i.e. that it is possible to build two towers of equal height, versus showing $f(n) = no$; i.e. that it is impossible to build two towers of equal height. In the former case, $f(n) = yes$, a demonstration of the building of the two towers is sufficient to prove the statement. This can be demonstrated by a drawing or one example, which Class C did for $n = 8$ and 7. In the latter case, $f(n) = no$, an example of the impossibility of the building two towers of equal height, will not be sufficient. The alternatives are either to show that no one of all the possible buildings of two towers will result in two towers of equal height, or argue theoretically for the impossibility. For $n = 10$ there are $\binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} + \binom{10}{5} = 637$ possible combinations of the building of two towers, and for $n = 9$, there are $\binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \binom{9}{4} = 255$. Evidently, several of those combinations will not give two towers of equal height, but there will be many combinations left that had to be explored. Practically it will be impossible or at least very difficult and also time consuming, to carry through those explorations. In reality the only possible way to determine that $f(n) = no$, is then to argue theoretically. Knowing that T_n is odd, it is trivial that $f(n) = no$, but Class C had not calculated T_n, and besides that their knowledge of parity would most probably be limited. Still, for $n = 10, 9, 6,$ and 5 Class C argued theoretically for the impossibility of building two towers of equal height. For the actual numbers they build two towers of nearly equal heights. To be more precise the difference between the two heights was one. Then they concluded: “No it is not possible. It is 1 in difference” (Class C, p. 1). As shown in section 6.2.1 this conclusion is correct since the difference, 1, is an odd number.

It is interesting to observe that Class C's argument for $f(3) = yes$ is based on another strategy than comparison. In this case they argued:

> 3 RODS ARE 6 LONG 6 CAN BE DIVIDED in 3+3 This gives 2 towers of equal height (Class C, p. 4)

As seen from the quotation above they used that $T_3 = 6$ can be divided in two equal parts, which in fact means that they argued that the building is possible since 6 is an even number. The fact that children may use different strategies, one for small numbers and another for larger numbers, when they work on the same task is also mentioned by Hughes (1986).

Class C gives no proof or explanation for the correctness of the discovered rule, beyond what they observed for the first 11 values.

6.3.1 Summary – Class C

Exploration of a special aspect of triangular numbers, T_n, was the main objective of this task. This exploration involved a variable, *n*, and a function in one variable, *f*.

For $n = 3$ this class calculated the value T_3, and determined the value of $f(3)$ using the evenness of T_3. For $4 \le n \le 11$, Class C determined the value of $f(n)$ without calculating the T_n value. The value $f(n) = yes$ is demonstrated by drawings showing two towers of equal height. These drawings demonstrate also indirectly the evenness of T_n. Their argument for the value $f(n) = no$ rests on the oddness of the number 1.

In addition this class has applied a recursive given relationship in an inductive manner, which means that aspects of the induction-principle have been implemented.

Mathematical structures inherent in this solution procedure:

- Explicit defined function;
- Figurative numbers (parity of numbers);
- Recursive defined function;
- Induction principle.

6.4 Class D

The solving strategy applied by this class can be described by the construction of appropriate tables which was investigated and based on these investigations relationships were stated.

Attached to this answer was a set of ten squares made out of cardboard. The smallest square was a 1×1-square, the biggest a 10×10-square. It appears not from the answer the role played by these squares in the class's investigation.

Compared with the answer from Class C, this answer is, as could be expected, a more formal one. Number symbols and mathematical terms have been used to a far greater extend than in the answer from Class C. The argumentation to Class D for the possibility or impossibility to build two towers with the prescribed property is based on the parity of T_n. If T_n is an odd number it is not possible, if T_n is an even number it is possible. For instance they argued at page 1:

> a) 55 is an odd number therefore it is impossible to divide in 2 equal parts.
> c) 36 is an even number. Therefore it is possible.

Mathematically, the separation of T_n into even and odd numbers is a grouping into equivalence classes given by the relation congruent modulo 2.

For the values $n = 1$ to $n = 13$ the results were organised in a table, which directly gave the relationship $f(n) = \begin{cases} yes \\ no \end{cases}$, where *f* is identical with the function (6.3) defined at page 98. In addition the *yes* values were circumscribed by red rectangles and the *no* values by blue rectangles. The pattern discovered in this table was shortly described "The pattern two at a time" (Class D, p.2). The meaning of 'Two at a time' had to be two *no*'s, two *yes*'s, two *no*'s, two *yes*'s, etc.. The reasonableness of this interpretation is supported by the circumscribed red and blue rectangles. The table was then extended to the value $n = 20$, and by applying the rule of two no's, two yes's, the value $f(20)$ was determined. Class D did not in their formulation of this rule emphasise that the pattern continues; i.e. goes on. But their use of the rule, in determining the value of $f(20)$, supports that this 'goes on' had to be tacitly understood. A mathematical description of this pattern is given by (6.4). As for Class C this, 'two at a time', could be related to apply an inductive principle on a recursive structure.

At the end of their answer the following table is presented:

Figure 6.3. Facsimile of the second table constructed by Class D (p.2)

The first sentence in this facsimile reads in English: It is only possible when it is even number(s).The second one: The number of blocks had to be in the 4 table, or one less. Mathematically the above table gives information or relationships between n, T_n and $f(n)$. The first column is the *n* values, the second the triangular numbers, T_n, and the third is the $f(n)$ values marked with the coloured rectangles, red for *yes*, and blue for *no*. Basis for the classification, red or blue rectangles was, as mentioned elsewhere, the parity of T_n, as they stated "It is only possible when it is even number(s)." The double arrows that connect some of the *n*-values, located at the left side of the table, are important and had to be noticed. In fact they can be interpreted to be a fourth column in the table. The question is then: What does the arrows mean or what about the interpretation of this fourth column? A closer look reveals that the double arrows connects the numbers in the 4-table, or a number in the 4-table with the preceding one.

Class D was convinced that it was possible to build two towers of equal height when T_n was an even number. The quotation, "It is only possible when it is even number(s).", emphasises that. In order to apply this rule the parity of T_n had to be determined, which meant that the value of T_n had to be calculated. This can be characterised as an indirect rule, and besides that the calculation of T_n without using a formula, is time consuming, and it seems that Class D was not familiar with such a formula. Then they may have raised questions like: Is it possible to find another and more direct rule to determine the parity of T_n?

or formulated as:

For which values of *n* is it possible to build two towers of equal height?

or in a mathematical language:

For which values of *n* will $f(n) = \begin{cases} yes \\ no \end{cases}$?

Most likely their solving procedure can be divided into five separate steps:

1. Firstly they wrote down the numbers 1,2,3,..., the *n*-values;
2. Then the T_n-values were calculated;
3. On the basis of the parity of T_n, the *yes*/*no* question was answered, the red and blue rectangles; i.e. the value of $f(n)$ was determined;
4. Then the double arrows were drawn, i.e. the *n*-values for which $f(n) = yes$, the red rectangles, were indicated by the arrows;
5. Lastly the double arrow pattern was explored and the conjecture was put down.

It should be noticed that the T_n-values was not calculated for $n = 13$ to 16, while the double arrows also includes $n = 15$ and 16, which were marked with red rectangles. The 'two at a time' rule told that those two, $n = 15$ and 16, were *yes* numbers. This solving procedure can be illustrated by the figure:

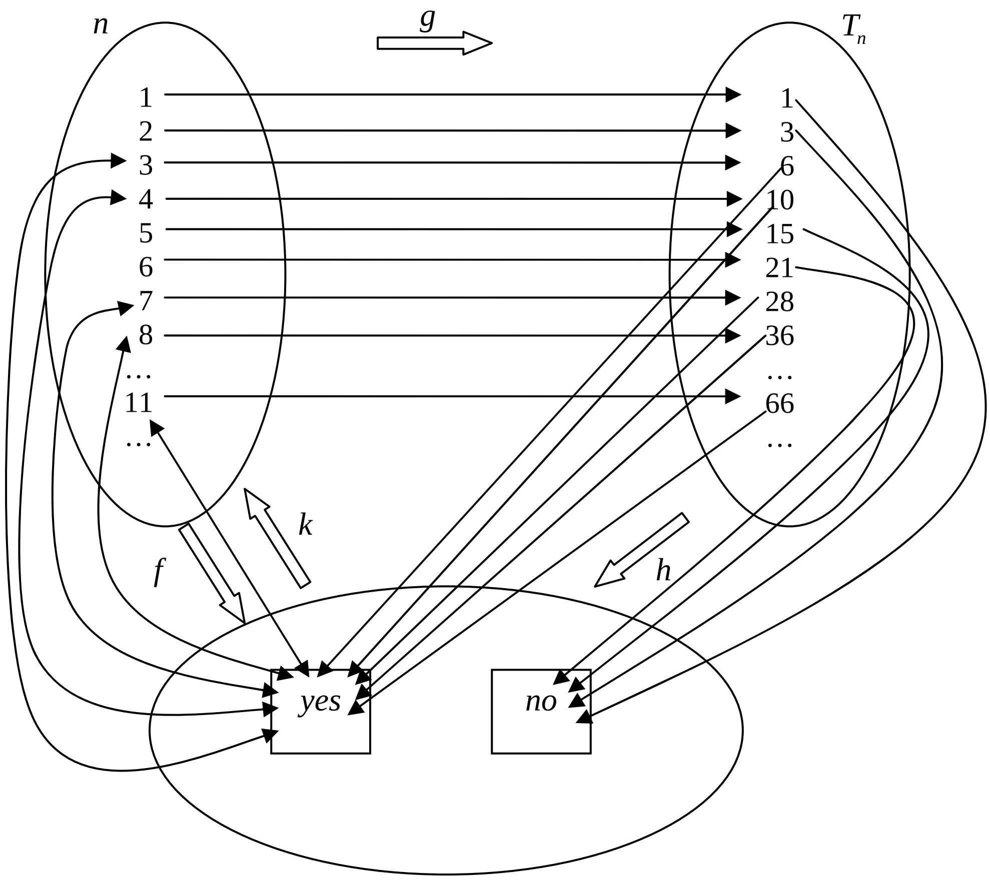


Figure 6.4. Illustration of the solving procedure to Class D

Here *g* symbolises the calculation of T_n, $g(n) = T_n$, and

h the calculation on the basis of the parity of T_n, $h(T_n) = \begin{cases} yes \text{ if } T_n \text{ is even} \\ no \text{ if } T_n \text{ is odd} \end{cases}$, and

k the drawing of the double arrows between some of the *n* values, and

f is the earlier defined function, $f(n) = \begin{cases} yes \text{ if arrow} \\ no \text{ if no arrow} \end{cases}$

g and *h* are functions, *k* is not a function because this relation is not unique.
A mathematical description of this solving procedure can be. In order to determine $f(n)$ Class D first applied the function *g*, and on the outcome, $g(n)$, they applied the function *h*. This can be symbolised as $f(n) = h(g(n))$ or $f = g \circ h$. The objective to Class D was a direct rule for *f*, for that purpose they classified the *n*-values in two categories, a *yes*-category and a *no*-category. Their explorations of these categories concluded with the rule: “The number of blocks had to be in the 4 table, or one less.”(Class D, p. 2). Indirectly this rule also states when it is impossible to build two towers of equal height. In a symbolic mathematical language this can be formulated:

$$f(n)=\begin{cases} yes & \text{if } n=4k \text{ or } n=4k-1 \\ no & \text{if } n\neq 4k \text{ and } n\neq 4k-1 \end{cases} \quad \text{where } k \text{ is a natural number} \qquad (6.5)$$

This is a direct and explicit rule, which is also correct, see section 6.2.3.

The rule (6.5) classifies the numbers *n* as *yes*-numbers or *no*-numbers according to their residue modulo 4. In a mathematical language; the *n*-values has been separated into four equivalence classes, the equivalence relation is the congruent modulo 4 relation. The *yes*-numbers consists of two equivalence classes; the numbers *n* that are congruent to either 0 or 3 modulo 4. For the *no*-numbers the two equivalence classes is defined as the numbers *n* that are congruent to either 1 or 2 modulo 4.

6.4.1 Summary – Class D

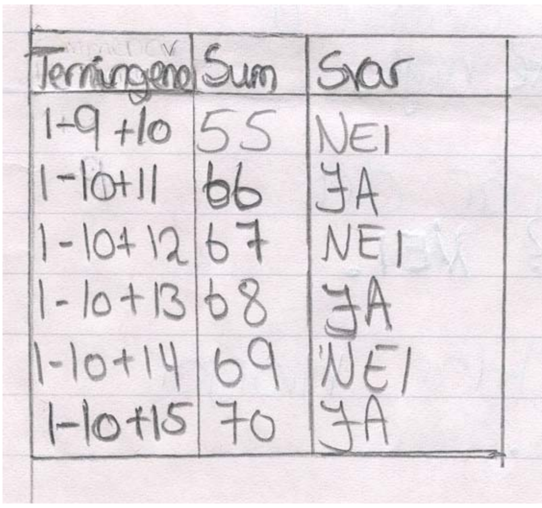
In order to determine if it was possible to build two towers of equal height, the parity of T_n played a crucial role, if T_n was an even number it was possible, if T_n was an odd number it was impossible. One outcome of the exploration of the T_n-numbers was a recursive relationship, which was applied in an inductive way. Their solution made use of functions and composition of functions. Another conclusion arrived at was an explicit formula for when it was possible to build the twin towers. Inherent in this solving procedure was the separation into equivalence classes, defined by the relation congruent modulo 4.

Mathematical structures inherent in this solution procedure:

- Functions and the composition of functions;
- Recursive defined function;
- Explicit defined function
- Figurative numbers;
- Equivalence relations;
- Equivalence classes;
- Induction principle.

6.5 Class E

In order to decide whether it was possible or not to build two towers of equal height, Class E investigated the parity of what they called “the sum of all the cubes” or “the sum of the cubes” (Class E, p. 2). At page 2 they summarised their results in the figure below:



Terningene	Sum	Svar
1-9 +10	55	NEI
1-10+11	66	JA
1-10+12	67	NEI
1-10+13	68	JA
1-10+14	69	NEI
1-10+15	70	JA

Figure 6.5. Table constructed by Class E (p. 2)

Translation of the Norwegian words: ‘Terningene’ means the cubes, ‘Sum’ is sum, ‘Svar’ is answer, ‘Nei’ is no, and ‘Ja’ is yes. It is observed from this facsimile that Class E interpreted the task differently from the two previous classes. The second and third lines in figure 6.5 deal with the values T_{10} and T_{11}. The next four lines handle with numbers calculated as $T_{10} + n$, where $n = 12, 13, 14, \text{and } 15$. This is not an interpretation of the given task that is unreasonable. The text in the task was: What about the building of two towers of the equal height if we in addition to the ten original cubes get a cube with 11 cm edges? Investigate this problem for other numbers and try to find a rule/ a pattern.
This class had most likely interpreted this as there was a ‘basis set’ consisting of the ten different cubes. Then they should investigate if it was possible to build two towers of equal height using this basis set and an 11-cube. The text did not underline that this 11-cube should be added to the ‘basis set’ when they investigated for $n = 12$ etc. This interpretation implies that S_n is an even number when *n* is an odd number. At page 3 Class E states that the building of the two towers is practicable if “the sum of the cubes” is an even number and impracticable if it is an odd number. From figure 6.5 it is observed that the pattern is a repeated ‘no, yes’ cycle, and that S_n is an even number when *n* is an odd number. This was however not stated by Class E. Why they did not state this relationship is impossible to trace in retrospect. It is noticed that for $n > 55$ it is not possible to build two towers of equal height, there will not be enough cubes. However, for $n = 20$, which implies that S_{20} is an odd number, they stated that the building was possible, but for this particular question the task specified that there was 20 different cubes, which the class has emphasised (p. 2), and in this case the sum in question, T_{20}, is an even number.

Related to the focus of this work, mathematical structures inherent in the solution procedures used by pupils in open-ended tasks, this interpretation of the task is of no importance for the analysis. Like Class D the table in figure 6.5 indicates a strategy which rests on functions and composition of functions. The solving procedure to Class E can be illustrated by:

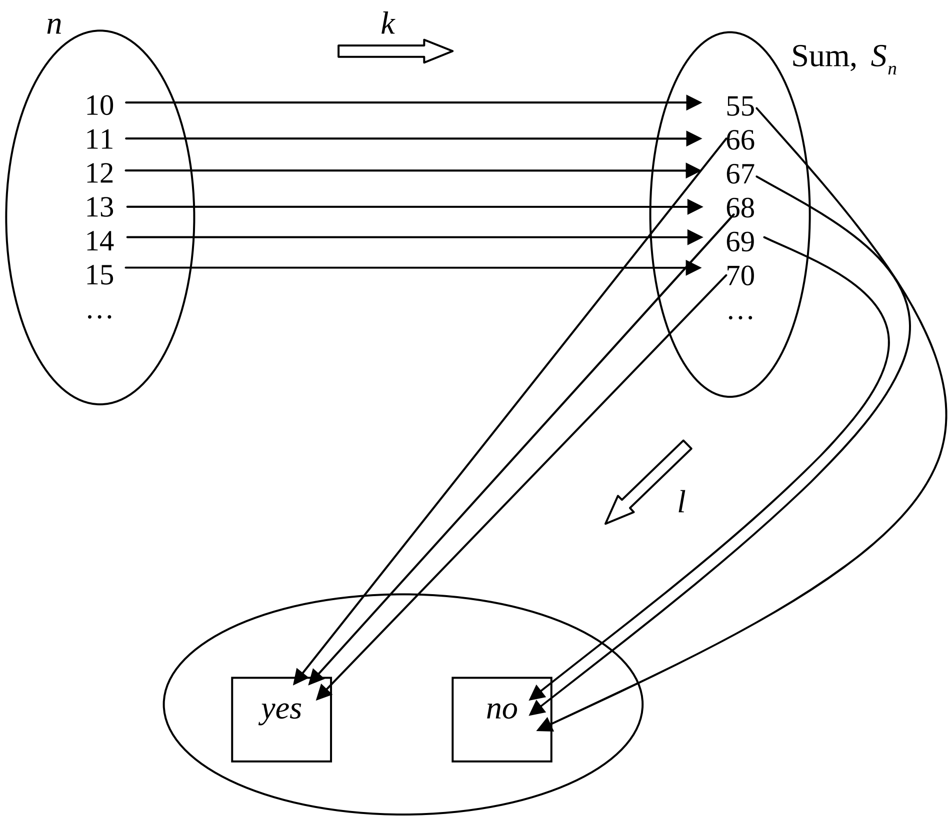


Figure 6.6. Illustration of the solving procedure to Class E

This shows that Class E had operated with:

- an independent variable *n*, which is the side length of the edge of the biggest cube;
- a function *k*, given by the formula $k(n) = S_n = T_{10} + n$;
- a function *l* which give the parity to S_n.

The solving procedure as illustrated in figure 6.6 can then be symbolised:

$$(k \circ l)(n) = l(k(n)) = l(S_n) = \begin{cases} yes & \text{if } S_n \text{ is an even number} \\ no & \text{if } S_n \text{ is an odd number} \end{cases} \qquad (6.6)$$

Compared with the solution to Class D this is a less extensive solution. Formula (6.6) is an explicit formula, but it is not what may be characterised as a direct formula for deciding the possibility of building the two towers of equal height, which in this context means that the value $(k \circ l)(n)$ is not directly based on the value *n*, but the value S_n.

6.5.1 Summary – Class E

Class E interpreted the task in a way that was not in accordance with the intended interpretation, it is emphasised that their interpretation is not unreasonable. Related to the focus of the analysis it is of no importance that they have an alternative interpretation of the task. The possibility of the building of the twin towers is linked to the parity to "the sum of the cubes", ending up with what can be classified as an indirect explicit formula. In their solving procedure they also applied functions and composition of functions.

Mathematical structures inherent in this solution procedure:

- Functions and compositions of functions;
- Explicit defined function;
- Parity of numbers.

6.6 Class F

> When the sum of all the cubes becomes an even number is it possible to construct two towers of equal height. When the sum becomes an odd number is it not possible. (Class F, p.1)

Class F gives with this double rule necessary and sufficient conditions for building two towers of equal height. It would, however, be more correct to say that they gives two sufficient conditions, one for the building of two towers of equal height, and one for not being able to do this building. Their argumentation for not being able to do the building of the twin towers was based on property of odd numbers.

In addition to the rule, quoted above, the possibility of building the twin towers was illustrated by a few examples. For example for $n = 8$ they stated:

Tower A: $2\text{cm} + 4\text{cm} + 5\text{cm} + 7\text{cm} = 18\text{cm}$
Tower B: $1\text{cm} + 3\text{cm} + 6\text{cm} + 8\text{cm} = 18\text{cm}$

As for the previous classes the above rule can be expressed as the relationship $f(n) = \begin{cases} yes \text{ if } T_n \text{ is even} \\ no \text{ if } T_n \text{ is odd} \end{cases}$, where *f* is identical with the function (6.3) defined at page 98. If the value T_n is known it is straightforward to apply this rule, and Class F gives a procedure or a formula for calculating this value.

> In order to calculate the sum of all the cubes together we add the first and the last number and multiply this number with the last number in the series divided by two. (Class F, p.1)

Translated into a mathematical symbolic language this can be coded:

$$T_n = (1+n)\cdot(n:2) \tag{6.7}$$

This rule is illustrated by two examples, $n = 4$ and $n = 10$, which demonstrates the correctness of the rule for these particular values. The way Class F demonstrates this rule is interesting. They wrote:

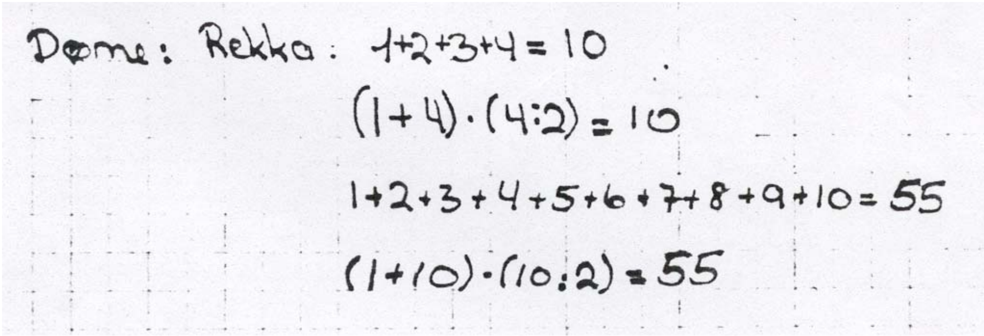


Figure 6.7. Explanation of the formula for T_n, Class F (p. 1)

Beyond these two examples there is no explanation or argument for the correctness of this rule. The way Class F presents the formula for T_n, makes it unlikely that Class F has found the rule in a formula-book. In such a book the formula would most likely be presented as $T_n = \frac{n(n+1)}{2}$ or $T_n = n(n+1)/2$, and both differ from the presentation given by Class F. The way the formula has been written gives a strong indication that Class F has observed that the sum of the first and the last number in the series $1, 2, 3, ..., n$ equals the sum of the second and the last but one number, etc., and that the total number of such identical sums are half of the numbers in a series. This is the same line of argument found in the well-known anecdote that is linked to Gauss. It could not be neglected that this anecdote was familiar for the class or for some of the pupils in the class. As mentioned in section 2.5.2 (page 40) the formula (6.7) could be regarded to be an expression for a function.

6.6.1 Extension of the task

One of the proposed extensions in the task was to investigate, for $n = 11$, how many different ways it was possible to build two towers of equal height. As has been pointed out in section 6.2.4 this is a partition problem with distinct summands and there is no, what may be called, simple formula that generates this number, which is 35. One strategy in determining this number is to list all the possible sums. That could be done by the pupils. Another strategy, but out of the scope for the pupils, is to apply generating functions, see section 6.2.4.

Class F has in their answer listed 11 of those 35 number-combinations. This list was organised in two columns, a 'Tower A column' and a 'Tower B column'. As has been emphasised in section 6.2.4 it is sufficient to determine the number-combinations in one of the columns, because when a number-combination in one column is settled, the corresponding number-combination in the other column is fixed. In this solution the 'Tower A column' was ordered very systematically. All the 11 number-combinations in the 'Tower A column' contained the numbers 11 and 10. The total number of twin towers that contains 11 and 10 in the same number-combination is 12. The missing number-combination in the 'Tower A column' is $(11,10,6,3,2,1)$. The ordering of the number-combinations in the 'Tower A column' makes it likely that the strategy used by Class F in their investigations, can be formalised in an algorithm. The following can describe the ordering of this column:

> Firstly write all the number-combinations which contains 9 as the largest number (after 11 and 10);
> then all the combinations which contains 8 as the largest number;
> then all the combinations which contains 7 as the largest number, etc.

Analysis of the 'Tower A column' indicates that Class F applied the following algorithm:

> Write down the number-combination that uses as many as possible of the largest numbers. In this case it is $k_1 = (11,10,9,3)$. This is the first number combination is given by Class F.

In order to construct a new number-combination apply either the procedure *F* given by:

> *F*: If possible split the smallest number in a combination in smaller addends. Do it in as many ways as possible;

or the procedure *G* given by:

> G : If possible reduce one of the numbers in the combination with one, and increase another number correspondingly with one, preferably, use numbers which are neighbours.

The two procedures *F* and *G* are not unique and are therefore not functions. Since 11 and 10 are members of all the number-combinations in the 'Tower A column', these two numbers are not affected by the procedure *G*. Class F numbered the number-combinations in the 'Tower A column' 1 to 11. These numbers are identical with the index numbers i to the number-combinations, k_i. Applying the procedures, *F* and *G*, described above, the following diagram for the ordering of the number-combinations in the 'Tower A column' can be made:

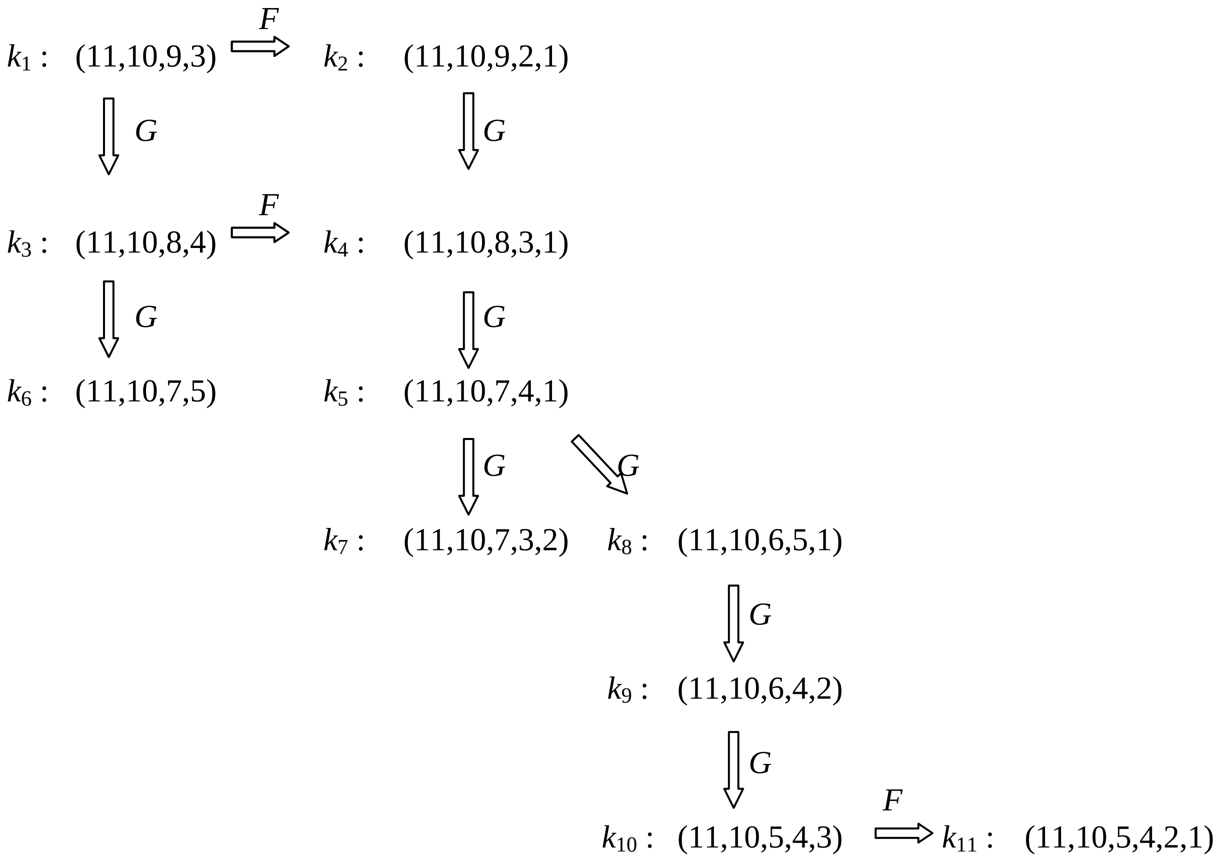


Figure 6.8. The ordering of the number-combinations in the 'Tower A column' related to the procedures *F* and *G*

As seen from figure 6. the 'Tower A column' begins with the number-combination $k_1 = (11, 10, 9, 3)$. Then the procedures *F* or *G* have been applied firstly on k_1, and then on the outcome of these procedures until the process ended with k_{11}. From the starting number-combination, k_1, one number-combination was deduced from a previous combination until the end, k_{11}. In a mathematical language this process can be described by:

> k_1 is given;
> For $i > 1$ and $j \geq 1$, k_i is given by either $k_i = F(k_{i-j})$ or $k_i = G(k_{i-j})$.

This is a recursive process involving the two recursive relations *F* and *G*.

It is remarked that the missing number-combination, $(11, 10, 6, 3, 2, 1)$, can be constructed by applying the procedure *F* on $k_9 = (11, 10, 6, 4, 2)$ where 4 is split in $3 + 1$. Another possibility is to apply the procedure *G* on the number-combination $k_{11} = (11, 10, 5, 4, 2, 1)$. In this case the increase of one is made on the number 5, and the decrease on the number 4; i.e. the increase is on the biggest number, and the decrease is on the smallest of the two changing numbers. This in contrast with the previous cases where the biggest of the two changing numbers has been decreased.

6.6.2 Summary – Class F

Parity of number and functions are central mathematical structures applied in this solution. The triangular numbers was defined by the explicit formula $T_n = (1+n)\cdot(n:2)$. In the construction of the number combinations for building the twin towers with height 33 it is argued that Class F has applied two different recursive procedures.

Mathematical structures inherent in this solution procedure:

- Figurative numbers (Parity of numbers and triangular numbers);
- Explicit defined function;
- Explicit defined relation;
- Recursive defined relation.

6.7 Class G

The use of the language in this solution makes it probable that this solution has been written by the teacher, see section 4.2.2, and since this solution was written with a typewriter it is impossible to use the handwriting as criterion for the authorship of this text. However, nothing in this text indicates that the teacher directly influenced or improved what may be called the mathematical quality of the product. Consequently this text can, as in the case of Class B, be regarded as a report of the achievements of Class G.

Whether or not it is possible to build the twin towers Class G linked to what they called *the sum of the height*. They claimed that if this quantity, which is identical with T_n, is an even number the building can be done, if not the construction is impossible. Their argument for the impossibility of the construction is based on the fact that an odd number is not divisible by two, and correspondingly an even number is divisible by two and the building of the twin towers can be done.

As suggested in the text of the task the pupils were encouraged to investigate the problem for different numbers *n*. The result of the investigation carried out by this class was presented in a table consisting of three rows. The first row was reserved for the number of cubes, *n*, the second row gave the T_n-values, and the third row kept order of the possibility for building the twin towers. This organising of the table indicates that the solving procedure has been the following: On the basis of the number of cubes the T_n-values were calculated and based on the parity of T_n the building problem was solved. The *n*-values and the answer to the building problem range from $n = 1$ to $n = 26$, while the T_n-values ends with $T_{20} = 210$. This solving procedure is to a certain extent identical with the solving procedure to Class D, and it can be illustrated:

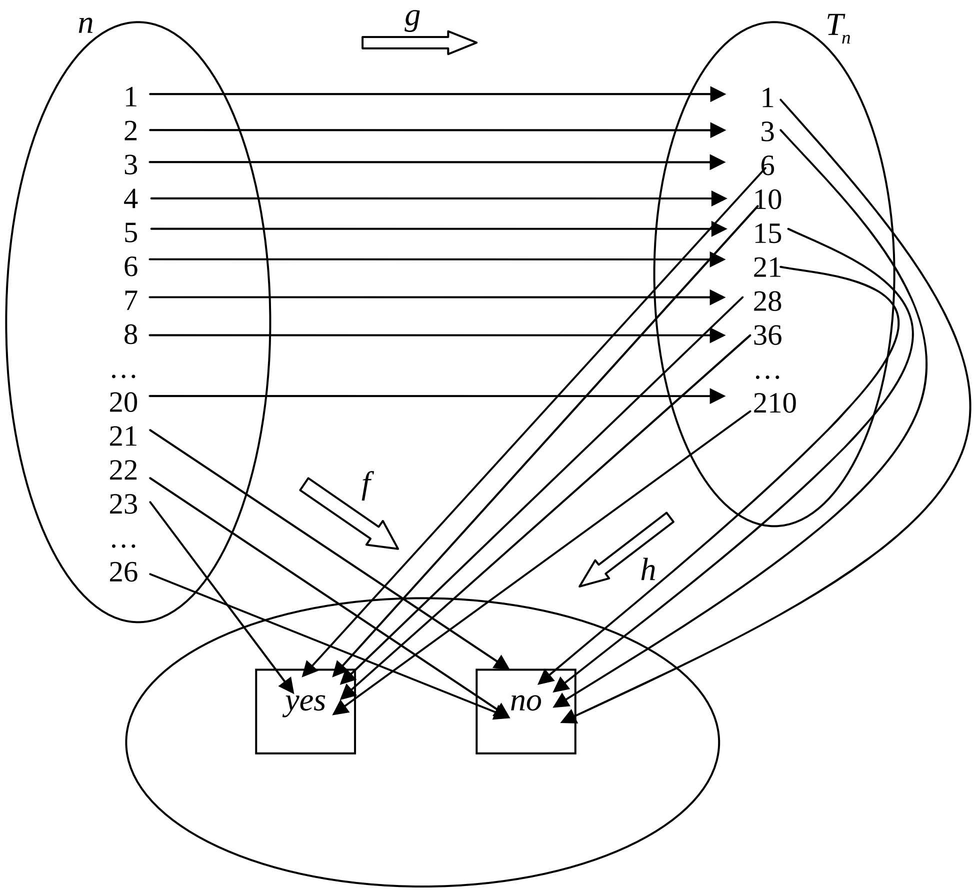


Figure 6.9. Illustration of the solving procedure to Class G

Here, as in section 6.4 the symbols in figure 6.9 are defined by:

g is the calculation of T_n; i.e. $g(n) = T_n$;

h is defined by $h(T_n) = \begin{cases} yes \text{ if } T_n \text{ is even} \\ no \text{ if } T_n \text{ is odd} \end{cases}$;

f is given as $f(n) = \begin{cases} yes \text{ if it is possible to bulid the twin towers} \\ no \text{ if it is not possible to bulid the twin towers} \end{cases}$;

where *g*, *h*, and *f* are explicit defined functions.

The solving procedure to Class G can then be symbolised $f(n) = h(g(n))$.

On the basis of the parity to the T_n-values the class stated that the pattern for the possibility of building the twin towers is given by "*no*, *no*, *yes*, *yes*, etc" (Class G, p. 1), which has to be understood as a claim that the pattern will continue, and will repeat itself with a period of four, and thus the pattern has a recursive structure (see section 6.2.2). This pattern is then used to calculate the $f(n)$-values for $n = 21$ to $n = 26$. In a mathematical language this pattern or rule is the same as described by (6.4).

6.7.1 Summary – Class G

The solving procedure applied by this class relied on the parity of numbers, functions and composition of functions. In the solution a pattern with a recursive structure is stated.

Mathematical structures inherent in this solving procedure:

- Figurative numbers (Parity of numbers);
- Explicit defined functions;
- Composition of functions;
- Recursive defined function.

6.8 Class H

An essay on the solution procedures to the task, *A set of ten cubes*, is a short, but apposite characteristic of this answer, which is focused mainly on what the pupils called rules and patterns, and in some instances also arguments for the correctness of these rules and patterns. All together this class presented four different rules or patterns. This answer does not, for any value of *n*, state values for T_n. But statements "the sum of the height to all the cubes" and "Examining the numbers with which we can build two equal towers (...)" (Class H, p.1) reveals they had a survey or a table giving the relation between *n* and T_n. Such a table can, as pointed out in the analysis for the previous discussed classes, be regarded as a function, *f*.

As mentioned in section 4.2.1 this class participated also in the competition with a solution of the task *Black and White Squares*. In a letter attached to the answer to that solution, the teacher (of this class) informed that when the pupils worked out the solution to this task, *A set of ten cubes*, they were not familiar with working algebraically. However, based on their results, and after they had mailed their answer he showed or demonstrated how this could be handled algebraically (Class R, p.1). This gives a strong indication that Class H here calculated the T_n-values as sums of natural numbers, and not by applying a formula.

The first rule settled was that the parity of T_n decides whether or not it is possible to build two towers of equal height. If T_n is an even number it is possible and if it is an odd number it is impossible. Their argument for the correctness of this rule was that an even number can be divided in two equal parts, which is impossible to do with an odd number. As done for the Classes C to G, this can be described by a function *f*. A mathematical description of this result can be $f(n) = \begin{cases} yes \text{ if } T_n \text{ even} \\ no \text{ if } T_n \text{ odd} \end{cases}$. They did not in this section of their answer give any examples of twin towers with equal height. In the suggested extension of the task, how many possible ways to build the twin towers when $n = 11$, they presented however several examples.

This class continued to investigate the problem, and they discovered a second rule that was formulated:

> (…) if you have an even number of cubes and divide by four, you can build to towers of equal height if you get an integer. Do you have an odd number of cubes you have to add one before you divide by four. But beyond that it is as for even numbers. (Class H, p.1)

Even though the formulation of this second rule, mathematically speaking, is not perfect, the decoding of this rule into a mathematical language entails no difficulties. Applying mathematical symbols it can be written:

If *n* is even, $f(n) = yes$ if $n = 4k$;
or if *n* is odd, $f(n) = yes$ if $n + 1 = 4k$ where *k* is an integer.

As known from section 6.2 this is correct. Their argument for this relationship was based on the following observation:

> The reason for why we have to divide by four is that between each odd number with which we can build two towers of equal height there are four numbers. Similar for even numbers. (Class H, p. 1)

The pupils examination of the T_n-values revealed that the pattern were four numbers between each odd index number, *n*, for which the building of the twin towers could be done, and similar for the even index numbers. More than that, they also observed that the odd indexes which fit into the pattern, were one less than a multiple of four, and that the even indexes were multiples of four. They did not explicit write that it is impossible to build two towers of equal height for the other values of *n*, but from their answer this is tacitly understood. This means that if the number of cubes, *n*, is odd, and $n+1$ is not divisible by four, or if *n* is even and not divisible by four, it is impossible to build two towers of equal height. In a mathematical language this can be written as:

$$f(n) = \begin{cases} yes \text{ if } n = 4k \text{ or } n = 4k-1 \\ \text{no if } n \neq 4k \text{ and } n \neq 4k-1 \end{cases} \text{ where } k \text{ is a natural number} \qquad (6.8)$$

which is an explicit rule.

The strategy underlying the solution procedure for the rule stated as (6.8), was a sorting of the numbers into two non-empty disjoint groups, the *yes*-numbers and the *no*-numbers. Based on a study of the *yes*-numbers this group of numbers was separated into two new non-empty disjoint classes, the residues 0 and 3 modulo 4. Consequently, the group of *no*-numbers were composed of the remaining numbers; i.e. the residues 1 or 2 modulo 4. This rule classifies the numbers as *yes*- or *no*-numbers according to their residue modulo 4. Mathematically Class H applied here a classification into equivalence classes. Using the congruent modulo notation expression (6.8) can be written:

$$f(n)=\begin{cases} yes \text{ if } n\equiv 0,3 (\text{mod } 4) \\ no \text{ if } n\equiv 1,2 (\text{mod } 4) \end{cases} \quad (6.9)$$

The solving procedure reviewed above is the same as described for Class D by figure 6.4, page 104, which means that composition of functions also is inherent in the solution procedure to Class H.

Class H continued their examinations of the T_n -values, and presented a third rule:

> Examining the numbers with which we can build two equal towers we found it was possible with each 3rd and 4th number. (Class H, p.1)

The procedure described in the above quotation is a 'counting' procedure, and has therefore a recursive structure; the preceding numbers defines the actual or next number. Compared with the second rule, which was an explicit rule, this has a different character. It is noticed that formulations like 'each 3rd and 4th (number)' is, in Norway, frequently used terms. In order to apply this rule, a starting value for the counting procedure is needed. This value is not emphasised by Class H, but it is reasonable to suppose that the starting value for the counting is 1. If the *yes*-numbers described by the given rule is called n_i , $i=1,2,3\ldots$ the first number, n_1, is 3, the second number $n_2=4$, the next is $n_3=n_2+3=7$, and $n_4=n_2+4=8$ etc.. This procedure can be described mathematically by:

$$\begin{aligned} &n_1=3 \text{ and } n_2=4 \\ &\text{For } i>1 \text{ apply } \begin{cases} n_{2i-1}=n_{2i-2}+3 \\ n_{2i}=n_{2i-2}+4 \end{cases} \end{aligned} \quad (6.10)$$

As for the second rule this rule treats the numbers modulo 4, and it lists all the numbers where the residue modulo 4 is 3 or 4 (i.e. 0). It is noticed that Class H in this rule only focus on the *yes*-numbers, the *no*-numbers are not mentioned. However, it is reasonable to interpret that the numbers which not are described as *yes*-numbers are the *no*-numbers. On that background this rule is also correct, and it is a variant of the no, no, yes, yes pattern described by Class C and Class D which mathematically can be expressed by (6.4).

However, Class H continued their explorations of the *yes*-numbers, and they discovered a particular pattern in the digits to those numbers, and a fourth rule was formulated:

> Continued examinations revealed that if the number at the tens-place is an even number we can build two towers of equal height if the number at the units-place is 0, 3, 4, 7 or 8. And if the tens-number is an odd number, the units-number must be 1, 2, 5, 6 or 9. (Class H, p.1-2)

It is noticed that Class H in their formulation of the rule did not use the corresponding Norwegian word for digit ('siffer'). Taken into consideration that in Norway the word for number ('tall') is frequently used as synonymous for the word digit, the interpretation of the terms 'the number at the tens place', and 'tens-number' causes no difficulties. The meaning of these terms had to be the tens-digit, and similar for the units-digit. Again, this is a correct rule, and it is valid without regard to the number of digits in the number.[10] This rule is, as the second rule, an explicit rule. This fourth rule classifies actually the numbers *n* as residues modulo 20. Decoded into a mathematical language the fourth rule can be formulated:

$$f(n) = \begin{cases} yes \text{ if } n \equiv 0,3,4,7,8,11,12,15,16,19 \pmod{20} \\ no \text{ if } n \equiv 1,2,5,6,9,10,13,14,17,18 \pmod{20} \end{cases}$$

Mathematically this is a separation of the numbers, *n*, into 20 equivalence classes.

An interesting question is: How did Class H discover this rule? If it is supposed that only two digits numbers are observed or investigated it is necessary to look at a table that goes as far as to $n = 29$, in order to go through the described pattern once. If one digit numbers are included in the observed numbers, this number is $n = 19$. If the table is extended to $n = 49$ the pattern will be repeated. To construct a table up to $n = 29$ existing of *n*- and T_n-values and where the T_n-values has been calculated as sums of natural numbers, is not an impossible task for a class. Doing the same to $n = 49$, is however another question. In their text they did not, as mentioned at page 114, refer to the T_n-values, but to the *n*-values. The actual and interesting values of *n*, are those, which make the building of the twin towers possible. Applying the second rule will relative quickly produce a table of the actual *n*-values. However, the easiest and also quickest way to construct such a table is to apply the third rule inductively. In order to produce an extensive table, it is probable to suppose that Class H has used the second or the third rule.

6.8.1 Extension of the task

As reported in section 6.2.4 there are, for $n = 11$, 35 different number-combinations which gives twin towers of equal height. Each twin number combination consists of two series of numbers, and each series has sum 33. Class H has written 16 of those twin number-combinations (Class H, p. 3-4). The number-combinations are numbered 1 to 16. One of the given combinations is wrong; in this case the two sums add to 34 and 32 respectively. The first third of these combinations can be characterised as unordered. The order of the number that make a number-combination are neither given in an increasing nor in a decreasing way; the order of the numbers are ‘higgledy-piggledy’. The last two thirds of the number-combinations are ordered, most of them in an increasing way. The way Class H has presented these number-combinations gives little or no indication of a systematic approach. The change from unordered to ordered number combinations, has most likely a practical background, it is easier to control that the number-combinations are different. This makes it probable that the strategy used by Class H best can be described as a ‘try and err’ or ‘search and find’.

[10] The residue modulo 4 is only dependant of the two lasts digits, the tens and units digits, in the number. This rule is also valid for a one digits number. The tens digit is in this case zero, which is an even number.

6.8.2 Summary – Class H

As for the other answers parity of numbers and function are central mathematical structures inherent in this solution procedure. Four different rules or relationships are presented, three of these state explicit rules, and one a recursive relationship. It is argued that this class in their solving procedure has applied a sorting of the numbers into equivalence classes. Residue modulo *n*, $n = 4$ and 20, are mathematical structures which also are inherent in this answer.

Mathematical structures inherent in the solving procedures to this class:

- Figurative numbers (Parity of numbers);
- Explicit defined functions;
- Recursive defined function;
- Equivalence relations (Residue modulo);
- Equivalence classes;
- Induction principle (most likely).

6.9 Class I

The solution to this class is a six pages document. The language and the handwriting reveal that it has been written by at least two different pupils. The first three pages, written in the *nynorsk* language, are kept strictly to answer the questions given in the task, and it deals not with the extension part of the problem. This part has a format or form that may be characterised as an answer to a traditional school mathematics problem. It is shortly argued for why the questions can be answered in the way chosen, and in addition the answer to each sub-question has been underlined by a double line. The last three pages, written in *bokmål*, focus mainly on the question concerning the number of twin towers that can be build. In addition this part gives a summary of most of the results that have been presented on the first three pages.

6.9.1 The main task

The parity of T_n was the first criteria used to decide whether or not it was possible to build the twin towers. It is stated that in order to have success in building the twin towers, the sum, T_n, has to be an even number (Class I, p.1). At page 5 they present an equivalent statement; "If it is possible to build two towers, is based on if half of the sum is an integer. $\left(\dfrac{x(x+1)}{4} = k\right)$".

Their answer contains two tables. The first table is presented on page 2. This table has two columns, one *n*-column and one *yes*/*no*-column, and it range from $n = 1$ to $n = 17$. The other

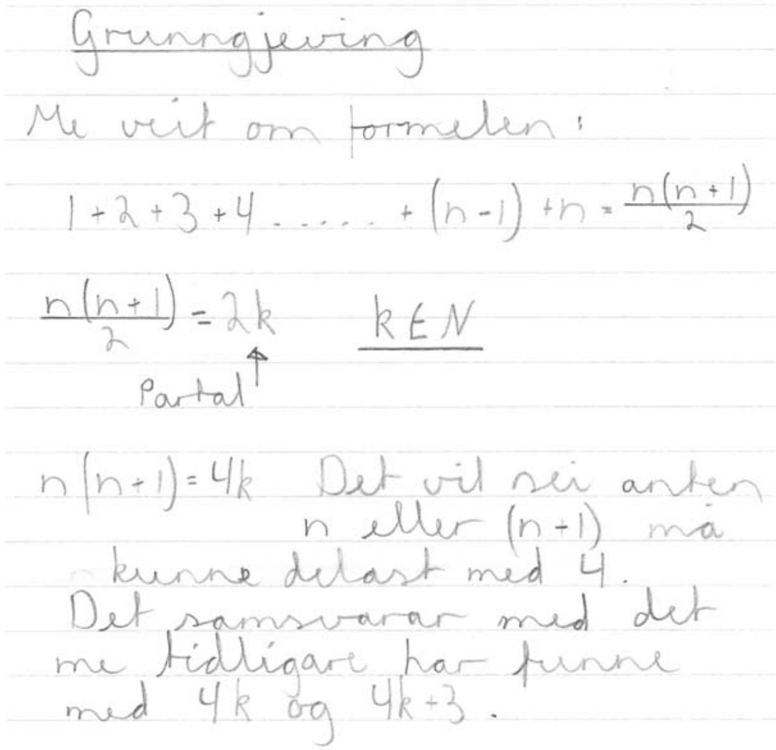
table is presented on page 4, and it has five columns, which are: n, $T_n, T_n/2$, a *yes*/*no*-column, and 'a number of twin towers' column. The table starts with $n=1$ and ends with $n=20$.
As for the previous classes these tables can be identified as functions.

The two-column table reflects the no, no, yes, yes pattern described by Class C, Class D, and Class H, and mathematically described by (6.4) (see the pages 99, 101, and 116). With reference to this table Class I concluded:

> We have examined the problem further and found this rule
>
> $$\boxed{4k \quad 4k+3} \qquad \underline{k \in \mathbb{N}_0}$$
>
> Has the biggest cube an edge which is an even number we use the formula: $4k$.
> Ex. biggest edge is 8, $4 \cdot 2 = 8$
> 8 cubes will make it.
> Has the biggest cube an edge, which is an odd number we use the formula: $4k+3$.
> Ex. biggest edge is 11, $4 \cdot 3 + 3 = 11$
> 11 cubes will make it. (Class I, p.2)

This is the same result as formulated by Class D and Class H, the difference is only the way the result is presented. Class I has used, to a far greater extent, mathematical symbols than the two other classes. The analysis done for Class D and Class H is therefore valid also for Class I, and as is emphasised in these analysis this rule classifies the numbers, n, according to their residues modulo 4. The two numerical examples, $n=8$ and $n=11$, underline that the building of the two twin towers is possible if and only if $n \equiv 0,3 \pmod 4$ where n is the length of the edge to the biggest cube. See pages 104, and 115 .

Contrary to the other answers, Class I gave a formal proof for the rule stated. Their proof was the following:



Grunngjeving

Me veit om formelen:

$1+2+3+4 \ldots\ldots + (n-1) + n = \frac{n(n+1)}{2}$

$\frac{n(n+1)}{2} = 2k \quad k \in N$

Partal ↑

$n(n+1) = 4k$ Det vil sei anten n eller $(n+1)$ må kunne delast med 4.

Det samsvarar med det me tidligare har funne med $4k$ og $4k+3$.

Figure 6.10. Class I's proof for the stated rule (p. 3)

The Norwegian text reads:

Proof

We know about the formula:

$$1+2+3+4.....+(n-1)+n = \frac{n(n+1)}{2}$$

$$\frac{n(n+1)}{2} = 2k \qquad k \in \mathrm{N}$$

Even number ↑

$n(n+1) = 4k$ This means that either n or $(n+1)$ had to be divided by 4.

This corresponds with what we found earlier with $4k$ and $4k+3$.

This is a correct proof for demonstrating that T_n is an even number if and only if $n \equiv 0, 3 (\mathrm{mod}\, 4)$. Their starting point was that the building of the two twin towers was possible if T_n was an even number and they knew the formula for T_n. This knowledge was translated to the mathematical expression $\frac{n(n+1)}{2} = 2k$, which then was simplified to $n(n+1) = 4k$; i.e. that the product $n(n+1)$ was divisible by 4. Using the fact that one and only one of the numbers n and $n+1$ is even, implied then that 4 had to divide either n or $n+1$, and the proof was complete. In a more detailed mathematical language this last arguments can be expressed as:

It is known that $n(n+1) \equiv 0(\text{mod}\,4)$, and that $(n,2) \neq (n+1,2)$. If $(n,2)=1$ then $n \equiv 0(\text{mod}\,4)$, and if $(n+1,2)=1$ then $n+1 \equiv 0(\text{mod}\,4)$; i.e. $n \equiv -1 \equiv 3(\text{mod}\,4)$.

It is remarkable to a surprising extent, that Class I carried through such a proof. Their proof reveals maturity in mathematical thinking. More than 30 years experiences in teacher education have told this researcher that reasoning like the above or similar, is difficult to understand for most of the teacher students. However, it could not be disregarded that this part of the answer to Class I, is work out by one pupil with special interests in mathematics.

6.9.2 Extension of the task

This part is introduced with a five column table, the second table in the answer. The first four columns of the table can be identified with the three first columns in the table presented by Class D, see page 102. As will be remembered, Class D investigated their table and then stated the modulo 4 relation, while at this point in the solving process Class I had, most likely, already stated the modulo 4 relation. Consequently there was no need for this class to include a column that was identical with the fourth column in Class D's table, instead they had a column called "different ways found" (Class I, p. 4). This column kept account for the number of twin towers that could be build for $n=3,4,7,8,11$.However, only the two first, the numbers for $n=3,4$, are correct. For $n=11$ it was stated that there are 27 different twin towers, but only two examples were given. Based on the available material, it is difficult to carry through further analysis. The sentence "Unfortunately we can not say that we found a system" (Class I, p.5), can possibly indicate a 'try and err' strategy.

6.9.3 Summary – Class I

Parity of numbers and functions are mathematical structures inherent in the solving procedures applied in this answer. Based on the *no*, *no*, *yes*, *yes* pattern, which describes a recursive rule, an explicit rule was formulated and also proved. Essential in the in the solving procedure was sorting into equivalence classes (residues modulo 4), triangular numbers, and divisibility properties of numbers.

Mathematical structures inherent in the solving procedure in this solution:

- Figurative numbers (Parity of numbers, triangular numbers);
- Explicit defined functions;
- Recursive defined function;
- Equivalence relations (Residue modulo);
- Equivalence classes;
- Divisibility of numbers.

6.10 Summary – A set of ten cubes

The pupils were guided relatively detailed through this investigation. They were encouraged to explore the building of the twin towers for different numbers. Mathematically this leads to explore the odd/even aspect of triangular numbers. The impossibility of a successful building process, that the triangular number was odd, was established by all the classes. Except for the grade 1 class, all the classes argued by directly stating that an odd number could not be divided in two equal parts, while the argument to the grade 1 class relied indirectly on the oddity. Correspondingly, all the classes, except the grad 1 class, took it for granted that the construction of the twin towers were practicable if the triangular number was even. The argument was that an even number can be divided in two equal parts. In some cases the building was exemplified. The grade 1 class demonstrated directly how the building could be carried out. Figurative numbers, more specifically odd/even numbers, played thus an important role in the solving procedures.

Another mathematical structure inherent in the solving procedures to all the classes was functions. These functions were defined by tables or as expressions. Most of the classes stated more than one expression, in that case both of the explicit type and of the recursive type. For some of the classes it is argued that they applied composition of functions.

For three of the classes it is also argued that inherent in their solving procedures is a sorting of the triangular numbers into equivalence classes, and based on the investigation of these equivalence classes the equivalence relation that defined these classes were established. One of the classes most likely applied two different recursive procedures when they explored in how many ways it was possible to build the twin towers using 11 cubes.

An overview of the mathematical structures inherent in the solution procedures to the classes is given in the following table:

Class / Mathematical structure	C	D	E	F	G	H	I
Recursive defined functions/relations	X	X		X	X	X	X
Explicit defined functions/relations	X	X	X	X	X	X	X
Induction principle	X	X				(X)	
Composition of functions		X	X		X		
Figurative numbers	X	X	X	X	X	X	X
Equivalence relations		X				X	X
Equivalence classes		X				X	X
Divisibility of numbers							X

Table 6.2 Overview of the mathematical structures inherent in the solution procedures to the task *A set with ten cubes*.

6.11 Remark

In issue **93**(6) p. 338, Oct. 1993, the journal *School Science and Mathematics* presented the following problem:

> Determine the set S of natural numbers n for which there exists integers $k_1, \dots, k_n \in \{0,1\}$, with $\sum_{i=1}^{n} (-1)^{k_i} i = 0$. For example, $2 \notin S$ because $1+2 \neq 0$ and $1-2 \neq 0$; $3 \in S$ because $1+2-3=0$.

This problem is in its nature identical with the task *A set of ten cubes*. The only difference is that the task presented in *Tangenten* put restrictions on the size of the number *n*, contrary to the problem quoted above.

It can be postulated with great credibility that a pupil in the compulsory school will not be able to read and understand the quoted problem, this in contrast with the task used in the *Tangenten* competition. With this in mind one remembers Goethe's word:

> Die Mathematiker sind eine Art Franzosen; redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anders. (Goethe, 1977, p. 455)

This exemplifies that the same idea can be represented by different language format, and that this matters didactically.

7. THE HUNDRED SQUARE

Three classes responded to this task:

Class J: A grade 3 class;
Class K: A grade 3 class;
Class L: A grade 4-6 class.

7.1 The task

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	15	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

This table contains the first hundred natural numbers.[11]
If we pick out a 3×2-rectangle from the table, for example

72	73	74
82	83	84

and multiply the numbers in opposite (diagonally) corners, $72 \cdot 84 = 6048$ and $74 \cdot 82 = 6068$, we find that the difference of their products is 20.

a) Pick out other 3×2 rectangles and carry through the same procedure.
What are your findings?

b) Explore this procedure for different type of rectangles; e.g.
- i) 4×2
- ii) 5×2

[11] In the original text in Tangenten there was a miss printing in this table, 29 replaced the numeral 39.

iii) 4×3

What are your findings in each of the cases?

c) Try to formulate a rule or a pattern (relation)!

d) Can you explain (prove) why your rule for the 3×2 rectangle is true?

This task can be extended in several directions; e.g.

I) Choose a figure that is not a rectangle.

What happens if we choose a cross like:

	25	
34		36
	45	

Multiply horizontally, 34·36, and vertically, 25·45, and find the difference between the products. What are your findings? Make larger crosses and carry out the procedure.

II) Change the table content. Use the multiplication table and carry out your explorations.

III) Organise the numbers (in the table) in a different way. What are the results of the preceding procedures if we put six numbers in each row:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18

etc.

A multitude of possibilities.

It should be noticed that the symbol × does not mean or indicate a multiplication operator In Norway the symbol for the multiplication operator is a dot (cdot), ·. The symbol, ×, is frequently, or usually, used to indicate the size, the dimension, of a rectangle, *length*×*height*. In the original Norwegian text the symbol x was used instead of the symbol × which is preferred in this text.

7.2 Solution

The text of the task is organised in such a way that it guided the pupils to carry through systematic investigations. These investigations had to be carried out in several directions, which mean that several independent variable quantities became an integral part of the task. Quantities as:

- *the position of the figures,* i.e. where in the table (hundred-square) these figures (rectangles/crosses) have been located;
- *the size (dimension) of the figures* and, eventually;

- *the number of numbers in the rows* i.e. a different format of the table.

The sizes of the figures, according to which figure used, depend on one or two independent variables. For a rectangle or a cross where length and height are different, the size is a variable of dimension two. The size of a square or cross where all of the arms are of equal length, is on the other hand a variable of dimension one.

7.2.1 Rectangle

It is suitable to introduce some symbols and terms. The complete process, picking a rectangle from the given table, and calculate according to the given algorithm, is called the *rectangle-algorithm* or shortly the *algorithm*. For this algorithm is used the symbol $R_{m\times n}(x)$, where *m* and *n* means respectively length and height of the rectangle, and *x* is the number which is located in the upper left corner of the rectangle. The variable *x* is then a variable for the position of the rectangle, and *m* and *n* for the size. If the table is the 10×10 hundred-square, both *m* and *n* are smaller than or equal to 10. In this case an arbitrary *m*×*n*-rectangle can be represented as:

x	$x+1$	....	$x+(m-1)$
$x+10$	$x+10+1$	....	$x+10+(m-1)$
. . .	. . .	....	. . .
$x+10(n-1)$	$x+10(n-1)+1$	....	$x+10(n-1)+(m-1)$

Figure 7.1 A *m*×*n*-rectangle from the 10×10 hundred-square.

The restriction $n \le 10$ is not important, and is due to the fact that the rectangle is placed inside the hundred-square. If the hundred-square is extended with natural numbers above 100, still organised with 10 columns, it is not necessary to put any restriction on *n*. Using the introduced symbols the algorithm can be symbolised:

$$R_{m\times n}(x) = (x+10(n-1))(x+10(m-1)) - x(x+10(n-1)+(m-1)) = 10(n-1)(m-1) \quad (7.1)$$

It is observed that the variable *x* is cancelled in the calculations. The position of the rectangle in the hundred-table is therefore of no significance for the outcome of the algorithm. This means that it is only the dimension of the rectangle that has significance for the answer. The symmetry of *m* and *n* in the expression (7.1) indicates that $R_{m\times n}(x) = R_{n\times m}(x)$, which can be shown to be correct. The above can be formulated as:

> The answer of the algorithm is always the increase in the first column of the rectangle, multiplied by the increase in the first row of the rectangle.

It is also noticed that $R_{m \times n}(x)$ always is a number in column number 10 and row number $(n-1)(m-1)$ in an extended table.

An extension of this task, mentioned in the text, was to organise the numbers in the hundred-square in a different way, and then carry through the algorithm. What happens if each row contains *t* numbers instead of 10? In this case an arbitrary *m*×*n*-rectangle can be described by:

x	$x+1$	....	$x+(m-1)$
$x+t$	$x+t+1$	....	$x+t+(m-1)$
. . .	. . .	....	. . .
$x+t(n-1)$	$x+t(n-1)+1$	....	$x+t(n-1)+(m-1)$

Figure 7.2 An *m*×*n*-rectangle from the hundred-square with *t* numbers in a row

In this case $m \leq t$. As was the case for $t=10$ it can be supposed that the number of rows in the table, consisting of the natural numbers where the rectangle is picked, is unlimited, so there is no need to put restrictions on *n*.

Comparing the above rectangle with the former one, figure 7.1, p. 126, it is observed that the only difference is that 10 is replaced by *t*. The result of the calculation is in this case:

$$(x+t(n-1))(x+t(m-1))-x(x+t(n-1)+(m-1))=t(n-1)(m-1) \qquad (7.2)$$

Again, and not unexpected, it is observed that the position of the rectangle is of no significance to the answer. The answer is depended of the number of columns in the original pattern, and of the dimension of the rectangle, and not of the position to the rectangle. The same rule as in the former case can be formulated:

> The answer of the algorithm is always the increase in the first column of the rectangle, multiplied by the increase in the first row of the rectangle.

In addition it is observed that the answer always is a number in the last column (column number *t*), and in row number (*n*–1)(*m*–1) in the table; i.e. the number at the end of row number (*n*–1)(*m*–1).

7.2.2 Crosses

Among the possible extensions mentioned in the problem text, was to choose other figures than a rectangle, and carry through similar algorithms on these figures. The text proposed for example a cross. Investigations on such figures were done by the classes. The task did not state that the crosses should have arms with equal lengths; in fact the pupils were encouraged to investigate figures with different shapes. One of the classes investigated also the case that the arms, in couple, were of equal length.

Crosses where all of the arms are of equal length

Analogous with the rectangle case, it is practical to introduce some symbols and terms.
A cross will be called an *n*-cross when the number of numbers (unit squares) from the centre of the cross, in each of the four directions is *n*. The cross exemplified in the task is then a 1-cross. This choice may look a bit strange, but the formulas become relatively simple with this measure for the crosses. According to this an *n*-cross has $length = height = 2n+1$ units (numbers). Placing an *n*-cross in a table with 10 columns restricts *n* to be less than 5.

The process of putting a cross in the hundred-square, and calculate according to the given algorithm, is called the *cross-algorithm*, or shorter the *algorithm* when there is no room for mistake. If *x* is the number in the centre of the cross the symbol $K_n(x)$ will be used for the outcome of this algorithm. As for the rectangle case, the variable *x* is taking care of the position to the figure.

An arbitrary 1-cross can be written as:

	$x-10$	
$x-1$	x	$x+1$
	$x+10$	

Figure 7.3 An *n*-cross from the hundred-square with 10 columns

The cross-algorithm gives

$$K_1(x) = (x-1)(x+1) - (x-10)(x+10) = -1 + 10^2 = 99$$

This tells that the answer, $K_1(x)$, is always 99. As in the rectangle case, the position of the 1-cross in the table is of no significance. If the algorithm is used on an *n*-cross, *n*<5, the answer is given by:

$$K_n(x) = n^2(10^2 - 1) = n^2 \cdot 99 \qquad (7.3)$$

which tells that:

The position of the *n*-cross in the table is of no importance, *n*-crosses of equal size give identical answers regardless of their position in the table.

With *t* numbers in the table rows an arbitrary 1-cross can be represented as:

	$x-t$	
$x-1$	x	$x+1$
	$x+t$	

Figure 7.4 An *n*-cross from the hundred-square with *t* columns

In this case the cross-algorithm gives $(x-1)(x+1) - (x-t)(x+t) = t^2 - 1$.
For an *n*-cross, $n < t/2$, the answer is $n^2(t^2 - 1)$.

Crosses where length and height are different

Only crosses that are symmetrical with respect to both a vertical axis and a horizontal axis will be handled. The reason for this is that one of the classes looked at crosses with such shapes.

Let *m* represent the number of squares on the horizontal arm from the centre of the cross to the end of one of the arms. Similarly let *n* represent the number of squares on the vertical arm, from the centre of the cross to the end of one of the arms. This means that *m* is a measure used to express the length of the cross, and *n* for the height of the cross. The total length of the cross is then $2m+1$, and the height $2n+1$. Since the length and the height of the cross is supposed to be different, $m \neq n$. Using a table with 10 columns, *m* has to be less than 5. As was the case using a rectangle it is not necessary to put any restrictions on *n*.

Let $K_{m\times n}(x)$ symbolise the cross-algorithm carried out on this type of crosses.

The algorithm gives now:

$$K_{m\times n}(x) = (x-m)(x+m) - (x-10n)(x+10n) = 10^2 n^2 - m^2 \qquad (7.4)$$

From (7.4) it follows again, that the position to the cross in the table, is of no significance, and more than that, $K_{m\times n}(x) \neq K_{n\times m}(x)$ since this expression is not symmetric in *m* and *n*. If the algorithm is used on an *m*×*n*-cross, in a table with *t* columns, $m < t/2$, the outcome of the algorithm is given by the formula $(tn)^2 - m^2$.

It can be noticed that it is possible to look generally at crosses. The case where the crosses have arms of equal length can be regarded as a special case of this one; i.e. where $m = n$ which mean that $K_n(x) = K_{n\times n}(x)$.

7.3 Comment on the task

Skovsmose (1998, 2000) has used this task as one example of an entry or a gate into what he calls *a landscape of investigation*. An idea to be investigated is: A geometrical figure is placed in the hundred-square, and a value is calculated according to a given algorithm. It is then observed that some figures behave invariant under translations in the hundred-square; i.e. the calculated value is constant regardless of the position in the hundred-square, and some figures did not have this quality. As shown in paragraph 7.2 rectangles and some crosses have this quality under the given algorithms. The question is then what happens if the algorithm is changed and/or if other geometrical shapes are introduced? What if negative numbers are considered? Calculating in other number system than base 10, will that influence the result? What about the result if the figures are rotated or if the number of columns is changed? There are a number of possibilities which Skovsmose (1998, 2000) summarised as: Consider any configuration of numbers $a_1, a_2, \ldots, a_n$ and any function $F = F(a_1, a_2, \ldots, a_n)$. What functions exhibit ‘nice’ properties under translation?

7.4 Class J

The solution to this class reveals that they acted strictly on the structure of the text to this task, which means that they restricted their investigations to rectangles. In their solution is only included the rectangular sizes mentioned in the text. Whether this class also investigated other rectangular sizes is impossible to decide in retrospection. The result of their investigation can be summarised as:

$$R_{3\times 2}(x) = 20, R_{4\times 2}(x) = 30, R_{5\times 2}(x) = 40 \text{ and } R_{4\times 3}(x) = 60.$$

Each of the statements was supported by three examples (Class J, p.1-2). That Class J just included three examples for each of the rectangular sizes, may look like it was ‘typographical’ motivated. Each of the three examples fitted nicely in on a page, see pages 1-2 in their answer.

As has been pointed out for other classes, for example Class A (page 80), it is reasonable to assume that the conclusion to Class J, in each of the cases, rests on more than the three examples written in their solution. The solution was a collaborative work made by the class, and it was the end-product of the collaboration, it is therefore very likely to suppose that the statements, mentioned above, rests on far more examples than written in their solution.

Common to all the rectangles referred to in this solution are that they are located (placed) with the first (top) rectangle row, on row number one or number two in the table. The number in the upper left corner of the rectangle, *x*, was in 9 out of 12 cases a number less than 10, in the three remaining cases it was a number between 11 and 20. Consequently, the factors in the products were relatively small. None of the multiplications, which the class had written in their solution, had an answer above 1000, the largest was $18\cdot 35 = 630$. This problem was solved by the class in the spring, before the first of June, at that time Class J was a grade three class. The mathematical syllabus in 1991 for a Norwegian grade three class included the basic multiplication facts, but not the multiplication algorithm. It would therefore be very unlikely that this class was familiar with the multiplication algorithm. One possibility is that they can have used a hand held calculator. We have no information about if the class based their calculations on technology, but concerning this study, that is not an interesting nor important question. The gist with this task was not to test the multiplication algorithm or the subtraction algorithm, but to look for relationships between the produced numbers.

On the basis of their results, Class J formulated:

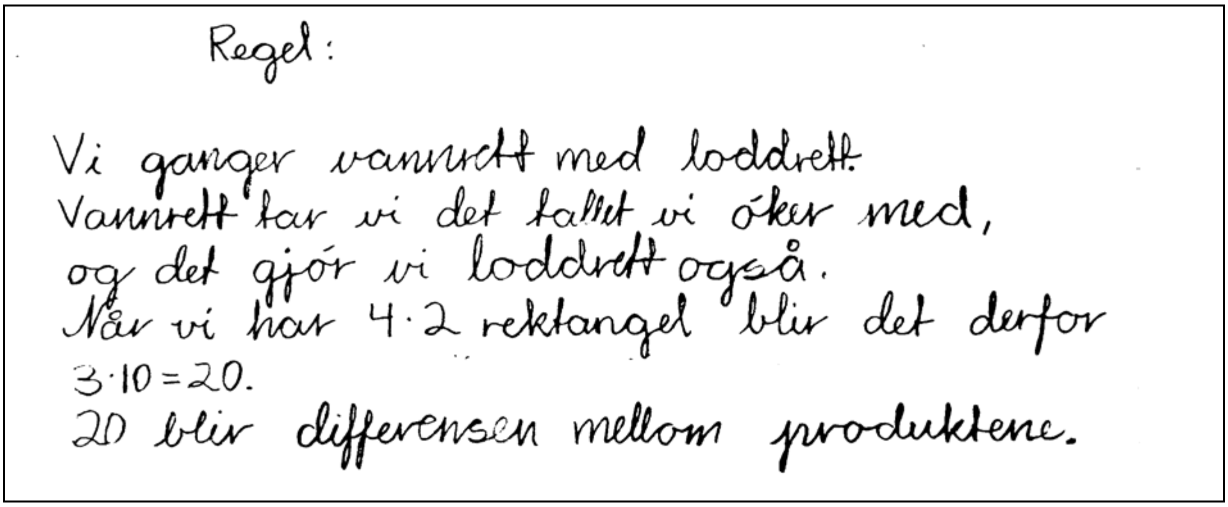

Regel:
Vi ganger vannrett med loddrett.
Vannrett tar vi det tallet vi óker med,
og det gjór vi loddrett også.
Når vi har 4·2 rektangel blir det derfor
3·10=20.
20 blir differensen mellom produktene.

Figure 7.5 Facsimile of the solution to Class J, p.3

The Norwegian text reads:

> Rule:
> We multiply horizontally by vertically.
> Horizontally we take the number we increase with,
> And that does we also vertically.
> When we have a 4·2 rectangle it becomes therefore
> 3·10 = 20
> 20 is the difference between the products.

As seen from the quotation the class has written, 3·10 = 20. Compared with what they wrote earlier, at p.1, where they stated that $R_{4\times 2}(x) = 30$, this is most likely a writing error, and as such, an error of no significance. It is noticed that Class J firstly has stated a general rule, which is explicitly formulated, and then the rule is illustrated with an example.

In the stated rule they have not explicitly emphasised that ‘rectangle of equal size gives identical answers’; i.e. the position of the rectangle in the table is without any significance for the answer, for the outcome of the algorithm. Most likely that is tacitly understood and accepted, since they claim that the result of the algorithm is only dependant of the size of the rectangle, and not of its position in the table. The size of the rectangle Class J described with the term “Horizontally … the number we increase with, and … also vertically” (p.3), accordingly length and height in the given rectangle. Using the mathematical symbols introduced in paragraph, (7.1), the rule stated by Class J can be presented as:

> The increase in the horizontal direction is $m-1$, in the vertical direction the increase is $10(n-1)$. Multiplying these two numbers gives $R_{m\times n}(x) = (m-1)\cdot 10(n-1)$.

This relationship is correct, and is identical with (7.1), page 126. This interpretation is supported by the example presented in the solution. For a 4×2 rectangle the increase in the horizontal direction is 3, which is $4-1$, and the increase in the vertical direction is 10, which is $10 \cdot (2-1)$. Related to mathematical structures this is a function in three independent variables, *x*, *m* and *n*, or what may be more in accordance with the mathematical terminology, a explicit function in one independent variable, *x*, and two parameters *m* and *n*.

A proof of the rule for a 3×2 rectangle, located with the top row in the first line of the table, was carried out by Class J in the following way:

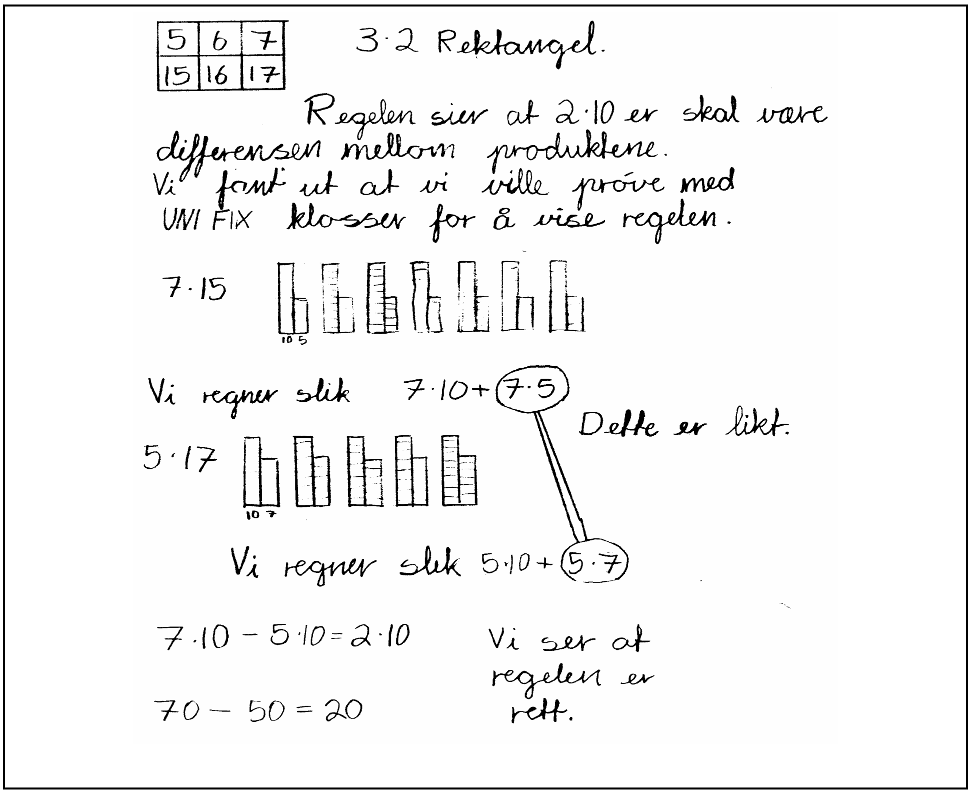


Figure 7.6 Facsimile of the proof to Class J (p. 3-4)

The Norwegian text reads:

(Figure) 3·2 Rectangle
The rule says that 2·10 is shall be
the difference between the products.
We found out that we would try with
UNI FIX cubes to show the rule.
7·15 (figure)
We calculate like this 7·10 + 7·5
This is identical.
5·17 (figure)
We calculate like this 5·10 + 5·7
7·10 – 5·10 = 2·10 We see that
the rule is
70 – 50 = 20 correct.

The way Class J executed their demonstration of the claimed rule is interesting. Firstly they ascertain that in this particular case the answer should be 2·10, then follows the argumentation (the proof). As seen from the quotation their proof is a mixture of number symbols and pictorial representations of numbers. Twice in their proof is found the structure:

number symbols $\rightarrow$ picture/UNI FIX cubes $\rightarrow$ number symbols

A calculation based on the same numbers, and that only rest on number symbols can for example be done in the following way:

$$7\cdot 15 = 7\cdot(10+5) = 7\cdot 10 + 7\cdot 5 \qquad (7.5)$$

It is necessary to write the factor that is greater than ten in an expanded form and then carry out a multiplication of a parenthesis. The picture, made by Class J, replaced the first identity in (7.5). This means that the original multiplication, $7\cdot 15$, was concretised as 7 fifteens. The second identity in (7.5) interpreted the picture in a different manner, it was interpreted as 7 tens and 7 fives, and this was written with symbols. Mathematically, the distributive law was used. Correspondingly for the multiplication 5·17. The two symbolic expressions, $7\cdot 10+7\cdot 5$ and $5\cdot 10+5\cdot 7$ was then compared. The 7 fives was now interpreted as 7·5 units, and the 5 sevens as 5·7 units. The commutative law for multiplication was applied, and therefore the two numbers, $7\cdot 5$ and $5\cdot 7$, was identical. The class was then left with the expression $7\cdot 10-5\cdot 10=2\cdot 10$, and concluded that the rule was correct. Mathematically they have used central characteristics to an algebraic structure called a *commutative ring*. The last sentence in their proof, $70-50=20$, is logically not necessary. Most likely has it been written in order to strengthen the conclusion. The line of argumentation carried out in this proof has a general nature (character). The class has used fixed number symbols, but in their argumentation Class J utilised only what may be called general qualities (characteristics) to the symbols. This is parallel to what is found in the solution to Class A, see page 82.

If $x<10$ a quite analogous argumentation can be used in order to calculate $R_{m\times n}(x)$. For growing heights of the rectangle the pictures get ‘bigger’ or the number of cubes increases, but beyond that it should cause no other difficulties. Using the commutative law for multiplication, the two products, corresponding with $7\cdot 5$ and $5\cdot 7$ in (7.5), will still be products where the factors are one-digit numbers, and hence it is directly observed that the products are equal.

If $x>10$ it is of course possible to illustrate the two multiplications using pictorial representations , cubes, of the involved numbers, but unlike the former case one cannot directly observe that the two products are identical. Proving the rule on this basis requires that the illustrations (pictures) or cubes must be adapted quite a lot. The number of cubes is large and the corresponding illustrations become relatively complex and therefore difficult to handle.

It is noticed that the numbers that are the basis for the proof is the same numbers that Class J used in the very beginning of their solution. The line of argumentation in the two cases is, however, quite different.

7.4.1 Summary – Class J

The investigations carried through by the class resulted in an explicit expression that mathematically is a function in three independent variables. The class demonstrated in a general way the correctness of this expression for the case $x = 5, m = 3$ and $n = 2$. In this demonstration Class J relied upon central results for a commutative ring.

This work gives also a nice illustration of how the pupils can alternate between mathematical symbols and pictorial representations of numbers. The pupils used both a symbolic language and a language based on pictures or UNI FIX cubes, and combined these two languages in a very elegant fashion.

Mathematical structures inherent in this solution procedure:

- Explicit defined function in three independent variables;
- Algebraic structure (Commutative ring).

7.5 Class K

This answer can be split into two parts. Part I, p.1-3, is the final solution to the task, the summary of their work. Part II, p.4-10, is what may be called ‘working pages’ or a draft, which to a certain extent shows how Class K worked when they were solving the problem. The content of these pages is the calculations related to the different rectangles and crosses investigated. If this is all the calculations carried through by the class is, however, uncertain. In any case that is probably without any significance for the analysis since the analysis is not directed towards calculation skills.

7.5.1 Rectangle

Like Class J, Class K did not in their answer consider any other rectangular sizes than mentioned in the text of the task. Their investigations can be summarised in the statements:

$$R_{3\times2}(x) = 20, R_{4\times2}(x) = 30, R_{5\times2}(x) = 40 \text{ and } R_{4\times3}(x) = 60.$$

Each of these statements was based on the calculation of many number examples. The different rectangular sizes used were placed (located) all over in the table, and also large numbers were included in the calculations. Most likely Class K used hand held calculators in order to calculate the multiplications. This supposition is mainly based on the fact that in their solution is only found the result of the multiplications, not the calculations that give these. This is in contrast to the subtractions, included in the rectangle-algorithm, where also the calculations are part of the report from the class. Another argument for asserting the use of hand held calculators is that Class K, like Class J, was a grade 3 class, and pupils at this grade would most likely be unfamiliar with the multiplication algorithm for numbers with more than one digit. See page 130.

On the basis of their findings Class K formulated this relationship "When the rectangle increases with one e.g. from 3·2 to 4·2 the difference increases by 10." (Class K, p.2). This relationship has a recursive structure. The result, the answer, of the algorithm carried out on a series of rectangles, is given by means of a previous result. This relationship opens up for different interpretations. According to the degree of stress that is put on the included number example, the various interpretations are more or less general. The crux of matter is what Class K meant with the term '*the rectangle increases by one*'. Isolated from the context this is a term, which is without any meaning or sense, but in this context it has several possible interpretations. What shall be *increased by one*?

- The *length*;
- the *height*;
- or *both*?

The *both* alternative that the length and height in the rectangle at the same time increases by one, is found to be very unlikely. The reason for this conclusion is mainly based on the way they have written the earlier mentioned results concerning the rectangles; going from one rectangle to the next only one of these two variables has increased.

The statement can be given a broad or very general interpretation; i.e. if little or no attention is attached to the given number example. The interpretation in this case is that the statement is valid for all types of rectangles when either length or height is increased by one. With the earlier introduced symbols this can be described by:
If the length of the rectangle is increased by one:

$$R_{(m+1)\times n}(x) - R_{m\times n}(x) = 10 \tag{7.6}$$

or if the height of the rectangle is increased by one

$$R_{m\times(n+1)}(x) - R_{m\times n}(x) = 10 \tag{7.7}$$

If the number example is stressed to a higher degree; i.e. the statement is meant to be only valid for rectangles with length 2, the interpretation of the statement is less general. With symbols:

$$R_{(m+1)\times 2}(x) - R_{m\times 2}(x) = 10 \tag{7.8}$$

Due to the following the interpretation (7.8) is found to be most likely. The interpretation (7.7) is very unlikely since Class K stated (p. 2) that $R_{4\times 2}(x) = 30$ and $R_{4\times 3}(x) = 60$, with 30 as the difference. The interpretation (7.6) is possible. From their answer it is not clear if Class

K has explored any other rectangular types than the four mentioned earlier, page 134. If they for example also had calculated $R_{3\times3}(x)$ they would most likely have observed or discovered that the difference between $R_{3\times4}(x)$ and $R_{3\times3}(x)$ is 20, and consequently that (7.6) was incorrect. However, on the basis of the presented number examples the interpretation (7.6) cannot be rejected, but it is hold as less likely than the interpretation (7.8). The main reason for this is that in three out of four presented number examples the rectangles were of the type $m\times 2$, the same type as exemplified in the given relationship. Regardless to which of these interpretations is the correct one, what is common for all of them is the recursive structure.

Class K stated also "All differences within the same type of rectangle are identical." (Class K, page 2). This statement is clear, and the interpretation is straightforward. The position of the rectangle is without any significance for the outcome of the algorithm; i.e. $R_{m\times n}(x)$ does not depend on *x*, the result is only depending on the size of the rectangle. In a mathematical language this means that rectangles behave invariant under translations in the hundred-square, which can be symbolised $R_{m\times n}(x) = constant$. This is correct, see page 126.

On page 3 Class K maintains that they did not manage to find or state any rule. This is an interesting remark because the statement to Class K concerning the rectangles is, mathematically speaking, a rule without any reference to which of the interpretations is the correct one. A natural question is then: Why did not Class K consider their statement or relationship to be a rule? Trying to answer this question in retrospect is a difficult or maybe an impossible task. An answer would therefore more or less have the status of a conjecture. One possibility could be that a rule had to be explicit; i.e. a recursive-structured rule is not considered as a rule, another possibility is that the formulated relationship was meant to be only locally correct; i.e. it was not a rule that was valid for all types of rectangles.

Class K has not stated any explicit relationship for $R_{m\times n}(x)$ when *m* and *n* are arbitrary natural numbers. The class has on the other hand stated explicit expression when $m = 3,4,5$, $n = 2$, and $m = 4, n = 3$ (see page 134).

7.5.2 Crosses

Class K followed one of the proposed extensions and investigated for crosses where all the arms had equal length. On the basis of their investigations they claimed "The differences become identical when the crosses are identical. Smallest cross this = 99. A bit larger cross this = 396" (Class K, p.3). The first claim is that this type of cross behaves invariant under translations in the hundred-square. Using the mathematical symbols introduced in paragraph 7.2.2 these relationships can be symbolised as:

$$K_n(x) = constant \quad , \quad K_1(x) = 99 \quad \text{and} \quad K_2(x) = 396$$

These relationships are, as shown in paragraph 7.2.2, correct. Class K did not explain or argue for these relationships beyond the number-examples presented in their answer.

7.5.3 Summary – Class K

For the rectangular shapes given in the text Class K found the correct values. Mathematically their explorations resulted in a recursive expression in three variables. It is argued that this expression is not an expression for an arbitrary rectangular shape, but most probably for rectangles of the form $m \times 2$. It is also noticed that Class K did not consider this expression to be a rule or a relationship.

Class K discovered that for the rectangles of the same size the result from the algorithm was independent of the position of the figure in the hundred-square, and correspondingly for crosses where all the arms had equal length. In a mathematical language, for these two types of figures, the figures were invariant under translations in the hundred-square.

Mathematical structures inherent in this solution procedure:

- Recursively defined function in three independent variables.

7.6 Class L

7.6.1 Rectangle

In their specification for the rectangle sizes Class L usually used the form *length*×*height*. However, on the pages 3 and 8 Class L specified the rectangle sizes by writing the height first; i.e. the dimension was written as *height*×*length*. In the analysis in this study the dimension is always given as *length*×*height*. That means what Class L on these two pages called an $n \times m$ rectangle is here called an $m \times n$ rectangle. The handwriting in the class report shows that various pupils have written this solution. This may well be the case for not being consequent in use of notations. For the further analysis, however, this discrepancy in the notations is without any significance.

Like the previous examined solutions Class L started with calculations of $R_{m\times n}(x)$, and they concluded (page 2-4):

$$R_{3\times2}(x) = 20, R_{4\times2}(x) = 30, R_{5\times2}(x) = 40, R_{6\times2}(x) = 50,$$
$$R_{3\times3}(x) = 30, R_{4\times3}(x) = 60, R_{5\times3}(x) = 80 \text{ and } R_{6\times3}(x) = 100.$$

The value for $R_{3\times3}(x)$ is not correct, correct value is 40. This can be a writing error. In this context the error is without any significance. Class L did not in their answer demonstrate the calculations for all the mentioned rectangle sizes. Like the two former classes, only the results from the multiplications or the final answer, the result of the algorithm, have been written.

The class did not, for an arbitrary rectangle size, explicitly state that $R_{m\times n}(x)$ is independent of *x*; i.e. rectangles of equal size give the same answer irrespective of the position in the hundred-square. That they still must have been aware of this fact appears indirectly from their given survey, where they stated what the answer was for an $m \times n$-rectangle. Further support for this is found at page 2 where they stressed that the answer is always 20 for a 3×2-rectangle. It is therefore likely to suppose that Class L was aware of the fact that rectangles were invariant under transposition in the hundred-square.

The class also stated "It becomes 20 in difference for 2×3 and 3×2." (Class L, p.3). That means they observed $R_{2\times3}(x) = R_{3\times2}(x)$. In a mathematical language it means that a 3×2-rectangle is invariant under a 90° rotation in the hundred-square, which is correct.

On the pages 4 and 5 is found a more detailed description of other patterns or relationships:

> For 3×2 is it 20 in the difference
> For 4×2 is it 30 in the difference
> For 5×2 is it 40 in the difference
> Each time you multiply it by 2 it becomes 10 in difference. (Class L, p. 4)

A similar statement is found for $m\times3$ –rectangles. In order to interpret these statements it is necessary to try to clarify the meaning of some of the terms or words used, this is primarily '*multiply it*' and '*difference*'. The sentence beginning with "Each time you multiply it by 2…", and the corresponding for $m\times3$–rectangles, may seem a bit turbid and the meaning is not obvious. In their text the pupils used the Norwegian word '*gangar*' which in a mathematical context has two different meanings.

I: Multiplication, 2×3, 2 times 3, (2 *gangar* 3);
II: To state the size of a rectangle, 2×3-rectangle, 2 by 3-rectangle, (2 *gangar* 3 rektangel).

The construction "Each time you multiply it by 2…" is referring back to the first three lines in the quotation which indicate that the word '*multiply*' in this context does not mean multiplication since these three lines does not deal with multiplication. Most likely Class L has in this context used the word '*gangar*' to state the size of the rectangle. Further support for this interpretation is the use of the first '*it*'-word in the sentence, which points to the rectangle. Besides, this is also in accordance with the usual Norwegian mode of expression to state the size of rectangles, *length* by *height*. The conclusion of this is that Class L has stated relationships for rectangles with height 2 or 3 respectively.

The word '*difference*' is here used in two different contexts. In the first three lines in the quotation the word refers doubtless to the answer of the subtraction part in the rectangle algorithm; i.e. to the result of subtraction between the two products which means $R_{m\times n}(x)$. In the fourth line Class L used this word for the difference between the answers of the rectangle algorithm; i.e. for example $R_{4\times2}(x) - R_{3\times2}(x) = 10$ and for rectangles with height 3 $R_{5\times3}(x) - R_{4\times3}(x) = 20$. This indicates the following interpretation of the quotation: For a rectangle of height 2, an increase in the length by one unit, will result in an increase of 10 in the output from the rectangle algorithm, or expressed by symbols

$$R_{(m+1)\times 2}(x) - R_{m\times 2}(x) = 10 \tag{7.9}$$

The corresponding result for rectangles with height 3 is

$$R_{(m+1)\times 3}(x) - R_{m\times 3}(x) = 20 \tag{7.10}$$

This interpretation established that Class L expressed two recursive relationships. This interpretation is further supported by what Class L writes on their next page:

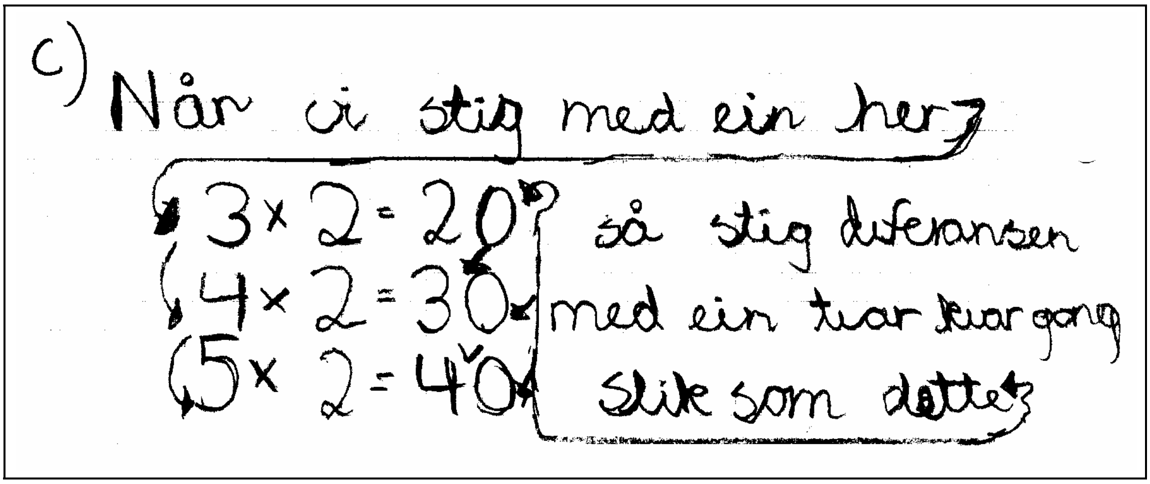


Figure 7.7 Illustrative explanation of the relationship (7.9)(Class L, p.5)

The Norwegian text reads:

When we increase by one here
3×2 = 20 then the difference increases
4×2 = 30 by ten each time
5×2 = 40 like this

Class L has here presented a general rule for the *m*×2 rectangles. The main support for this interpretation is the written formulation of the rule, and particular what is written at the end of the sentence, ‘like this’. The main purpose of the calling of attention to these three written number examples was, most likely, to make the explanation of the rule easier. In addition, the formulation ‘when we increase by one here’ is not restricted only to the cases $m = 3,4,5$, but has a general character. The interpretation of the presented rule is then, if the length of a rectangle with height two is increased by one, then the value produced by the algorithm (the difference) is increased by 10. This is identical with (7.9). Together with the starting value, $R_{3\times 2}(x) = 20$, stated by Class L, (7.9) gives a complete characterisation of the result for the rectangular sizes *m*×2. In a mathematical language this is a recursive relationship, or as it also is named, a first order recurrence relation.

The algorithm carried out in a table with six numbers in a row

The third suggested extension of this task was to change the number of elements in each of the rows in the table, and then carry through the algorithms. Class L grabbed this idea and investigated what happened if the rectangle algorithm was performed on a table with six numbers in the rows. In connection with the six number-examples presented by Class L in this part of their answer, is the statement:

> If you multiply 2×3 in both tables you find the answer at the end of the second row. If you multiply 2×4 in both tables you find the answer at the end of the third row. If you multiply 2×5 in both tables you find the answer at the end of the fourth row. If you multiply 2×6 in both tables you find the answer at the end of the fifth row. etc (Class L, p.8)

In this statement the word *multiply* is used in connection with symbols as 2×3, 2×4 etc. These symbols are here used as symbols for identifying the dimensions of the rectangles they do not denote the multiplications $2 \cdot 3$, $2 \cdot 4$, etc., see pages 125 and 138. Further support for this interpretation is the fact that the number 6, the result from the multiplication $2 \cdot 3$, is found at the end of the first row, while $R_{3\times 2}(x)$ is located at the end of the second row.

Correspondingly, the result from the multiplication $2 \cdot 6$ is found at the end of the second row, while $R_{6\times 2}(x)$ is found at the end of the fifth row. More than that, the result from the multiplication $2 \cdot 4$ is not found at the end of any of the rows in the table, while $R_{4\times 2}(x)$ is located at the end of the fifth row. The same argument is valid for $2 \cdot 5$. In this context (the Norwegian) the word 'multiply' refers therefore most likely to the multiplications in the rectangle algorithm. The interpretations of the sentence "If you multiply 2×3 in both tables you find the answer at the end of the second row", is then:

> If you carry through the rectangle algorithm with a 3×2 rectangle in both number tables, you find the answer at the end of the second row.

With symbols: $R_{3\times 2}(x) = 2t$ where t is the number of elements in a row, $t = 6 \text{ or } 10$. As the above quotation shows, Class L did not restrict themselves to look only at the 3×2 rectangle. Analogous statements were set out about other rectangles, all of them with height 2. In a mathematical language these statements can be described as:

$$R_{m\times 2}(x) = (m-1)t \quad \text{where } t = 6 \text{ or } 10 \text{ and } m = 3,4,5,6 \text{ etc.}$$

Even though Class L investigated for tables with 6 or 10 columns, their statement does not emphasise that it is only valid for that types of tables. The statement has a general form and the etc. at the end of the statement indicates that it is valid without any restriction put on t. This is correct, and it is the special case $n = 2$ of (7.2), page 127.

It is interesting to notice that in the first part of their answer Class L gave a recursive relationship for $R_{m\times 2}(x)$, (7.9), while Class L here has given an explicit relationship for the same quantity.

7.6.2 Crosses

Class L presented here the following rule "We have discovered if you increase the length with 2 squares then the difference increases with 300." (Class L, p.7). The rule was illustrated by two number examples that can be symbolised as: $K_{1\times 2}(66) - K_{1\times 1}(66) = 300$ and $K_{2\times 2}(66) - K_{2\times 1}(66) = 300$. As for the rectangle case Class L has also here used some words which needs a clarification, these are *length* and *difference*.

From the illustrations that goes together with the number examples referred above, it appears that *length*, in the Class L sense, corresponds with what, in paragraph 7.2.2, was called the height in a $m\times n$-cross, i.e. the n-value.

For the word *difference* the same considerations as stated for rectangles are valid, see page 138. The structure of the statement to Class L has the form: A fixed increase in one value causes a fixed increase in another value. If the n-value is increased by two units then the difference is increased by 300. In this context two subtractions have been carried out, one in the algorithm; i.e. the calculation of the K-value, and the subtraction that compares the two K-values. For the given number examples the differences between the two K-values are 300, it is not increasing by 300. What is increasing by 300 when the n-value is increased by two units, are the numbers produced by the algorithm. In a mathematical language the *first difference* between two subsequent K-values is 300. The investigations were carried out on figures dependent of both length and height. The stated rule is restricted to the case where the height varies and the length is fixed. The relationship can be symbolised:

$$K_{m\times(n+1)}(x) - K_{m\times n}(x) = 300 \text{ or } K_{m\times(n+1)}(x) = K_{m\times n}(x) + 300 \qquad (7.11)$$

Again it is noticed that Class L stated a rule where one value is given by means of previous known values. In a mathematical language this is a recursive relationship. It can be shown that (7.11) is correct only for the case that $n = 1$.

7.6.3 Summary – Class L

The explorations carried through by Class L included both rectangular shapes and crosses. Their investigations included several rectangular sizes, however all the shapes mentioned in their report were of the types $m\times 2$ or $m\times 3$. It is argued that Class L on the basis of their investigations stated two recursive relationships in two independent variables that were correct. Their investigations included also what happened if the number of columns in the table changed. Here they stated a correct explicit relationship for the rectangles of type $m\times 2$. For one type of crosses they maintained a recursive relationship, which was partly correct.

Mathematical structures inherent in the solution procedure:

- Explicit defined functions;

- Recursive defined functions.

7.7 Summary – The Hundred square

For all the classes, translation of the position of the different rectangles in the table did not make any difficulties for the pupils. The relationship for calculating the value of the rectangle-algorithm, (7.1), is an expression containing three independent variables, or maybe it is more correct to say that the expression containing one independent variable, *x*, and two parameters, *m* and *n*. It is argued that the classes discovered that rectangles behave invariant under transformation. Class K conjectured the same for crosses. The same class discovered also that rectangles behave invariant under a 90°'s rotation.

One of the classes, Class J, formulated an explicit rule, a statement. They also argued for, or proved, the correctness of this rule. This proof has a general character even though they used fixed number symbols. The proof relied upon central qualities of a commutative ring. The two other classes formulated statements that had a recursive structure. Beyond the number-examples they did not argue for the correctness of their statements. For Class K it is argued that the statement was restricted to rectangles of the form $m \times 2$.

Class L conjectured recursive relationships for the rectangular sizes $m \times 2$ and $m \times 3$, and an explicit relationship for the first of those two; i.e. for rectangles with height 2. This last statement was also given for tables where the number of columns was different from ten. All of these conjectures were correct. For the cross-figures Class L also made a conjecture that was a recursive relationship, however this conjecture was only partly correct.

An overview of the mathematical structures inherent in the solution procedures to the classes is given in the following table:

Class / Mathematical structure	J	K	L
Explicit defined functions	X		X
Recursive defined function		X	X
Algebraic structures	X		

Table 7.1 Overview of the mathematical structures inherent in the solution procedures to the task *The hundred square*

8. BLACK AND WHITE SQUARES

Six classes responded on this task:

Class M: A grade 2 class;

Class N: A grade 4 class;

Class O: A grade 5 class;

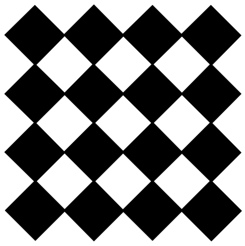
Class P: A grade 7 class;

Class Q: A grade 8 class;

Class R: A grade 8-9 class.

Class Q was from Sweden and was the only non-Norwegian class participated in the *Tangenten* competition. Two of the classes on this list had participated in one of the earlier competitions, Class N as Class D and Class R as Class H, see page 58.

8.1 The task



This pattern is made up of 25 black and white squares. It has seven squares across (where it is widest). You shall investigate similar patterns for different sizes. Examine the number(s) of squares in the patterns. Try to find a rule (a relation) between the width of the figure and the numbers of black and white squares in the pattern.

How many squares are there in a pattern that has 99 squares across? Can the class manage to solve this task in different ways? Discover the class other relationships when they work on the task?

The idea to this task was found in 'The Blue Box' (Shell Centre for Mathematical Education, ,1984).

8.2 Introduction

The relationship between the text and the given pattern is not the best; in fact there is a discrepancy. The phrase 'It has seven squares across (where it is widest).' is not in accordance with the illustration. A, what may be called standard interpretation of the pattern will claim that this pattern has four squares across, and not seven. It has seven squares at the diagonal, but not across. The reason for this discrepancy is that in the original text send to the editor, the pattern was located as shown in figure 8.1. The location in *Tangenten* was however as shown in figure 8.2.

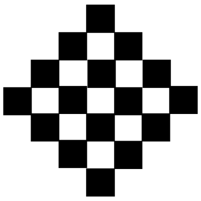


Figure 8.1 The intended localisation

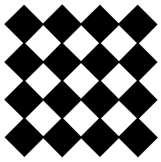


Figure 8.2 The implemented localisation

None of the participating classes commented on this discrepancy.

8.3 Solution

It appeared that the pupils had different interpretations of the text. It was the constructor's intention that the diagonals in a pattern should start and end in a black square. However, Class P constructed the patterns in a different manner, for every second pattern the diagonals started and ended in a white square. In the solution given here it is supposed that the diagonals always starts and ends in a black square. The consequences of Class P's interpretation will be considered when their solution is analysed.

There are several different solving strategies that can be applied on this task. The article in *Tangenten* which commented the pupils' solutions, described six different strategies (Torkildsen, 1991d). In this study the four solving-strategies used by the classes will be presented. There is however a multitude of solving-strategies that can be used, see for example Shell Centre for Mathematical Education (1984).

In order to describe the solutions in a mathematical language, it is appropriate to introduce some names and symbols. In the rest of this chapter the word *diagonal* means one of the two main diagonals in the quadratic pattern. The number of squares at the diagonal will be called *t*. A *t*-pattern is a pattern with *t* squares at the diagonal. The assumption that the diagonal in a pattern always starts and ends in a black square has as one consequence that the number of white squares at the diagonal is one less than the number of black squares at the diagonal. This implies that *t* is an odd number. Two succeeding patterns, a *t*-pattern and a (*t*+2)-pattern, for example a 3-pattern and a 5-pattern, are called *neighbouring* patterns. Further will

$B(t)$ symbolise the number of black squares in a *t*-pattern;
$W(t)$ symbolise the number of white squares in a *t*-pattern;
$A(t)$ symbolise the total number of squares (black and white) in a *t*-pattern.

The relationship between these quantities is $A(t) = B(t) + W(t)$.

Geometrically these patterns could be made up of triangles and/or squares put together by black and white squares. Algebraically this implies that special interest will be put on triangular numbers and square numbers. The symbols T_k will be used for the triangular number *k*, and S_k for the square number *k*. It turns also out that it is practical to have a symbol for the number of black squares on the diagonal in the *t*-pattern. If that number is called *n*, the number of white squares at the diagonal is $n-1$, and the relationship between *t* and *n* is then given by $t = 2n-1$. The next sections will present the four solving strategies.

8.3.1 Using a table

This strategy can be described by: Construct a table that gives an overview over the number of squares in the smallest *t*-patterns. Examine the numbers in the table and look for relationships. Below is the table for the nine smallest figures:

t	3	5	7	9	11	13	15	17	19
$B(t)$	4	9	16	25	36	49	64	81	100
$W(t)$	1	4	9	16	25	36	49	64	81
$T(t)$	5	13	25	41	61	85	103	145	181

Table 8.1 The number of black and white squares in the smallest patterns

From table 8.1 it is observed that the two rows in the middle nearly contains the same numbers, the exceptions are the first number in the $W(t)$-row, and the last number in the $B(t)$-row. This relation can be described by; the number of black squares in one pattern equals the number of white squares in the next pattern, with mathematical symbols:

$$W(t) = B(t-2) \tag{8.1}$$

That this is correct follows directly from the construction of two neighbouring patterns.

Another relationship observed in table 8.1: The sum of the number of squares on the main diagonal, *t*, and the number of white squares, $W(t)$, equals the number of black squares, $B(t)$. This relationship can also be formulated: The difference between the number of black squares, and the number of squares at the main diagonal equals the number of white squares, or alternatively: The difference between the number of black squares, and the number of white squares equals the number of squares on the main diagonal. With mathematical symbols this last relationship can be written:

$$B(t) - W(t) = t \tag{8.2}$$

This relationship is also correct since this difference is the difference between square number *n* and square number $n-1$, and this difference equals the odd number *n*, which is $2n-1 = t$. For an odd number $t > 1$, the two relationships (8.1) and (8.2) can be combined to:

$$W(t) = B(t-2) \text{ and } B(t) = B(t-2) + t \tag{8.3}$$

Applying these two expressions together with the relationship that the *t*-values increases by 2

$$\text{next } t = t + 2 \tag{8.4}$$

and the starting values $W(1) = 0$ and $B(1) = 1$, the numbers of squares in an arbitrary *t*-pattern, $A(t) = B(t) + W(t)$, can be determined. For the 99-pattern the number of squares is 4901.

8.3.2 Geometrical – Sum of two squares

The pattern of squares can be considered as two quadratic patterns interwoven in each other, one pattern with black squares, and one with white squares. The number of squares, at the side of the black pattern, is identical with the number of black squares at the diagonal in that pattern, i.e. $n = \frac{t+1}{2}$. For the second pattern, the white one, the number of white squares at the side, is one less than in the black pattern, i.e. $n-1 = \frac{t-1}{2}$. Combining these two facts give that the number of squares in a *t*-pattern can be calculated as:

$$A(t) = S_n + S_{n-1} = \left(\frac{t+1}{2}\right)^2 + \left(\frac{t-1}{2}\right)^2 \qquad (8.5)$$

which can be simplified to:

$$A(t) = \frac{t^2+1}{2} \qquad (8.6)$$

and thus $A(99) = \frac{99^2+1}{2} = 4901$. If *n* is used as the variable formula (8.6) can be written:

$$A(t) = A(2n-1) = n^2 + (n-1)^2 \qquad (8.7)$$

8.3.3 Geometrical – Four triangles and a square

If the pattern is rotated 90° from the position given in the text, it is easier to observe that the pattern can be decomposed into four triangles and a single square. An example is shown in figure 8.3.

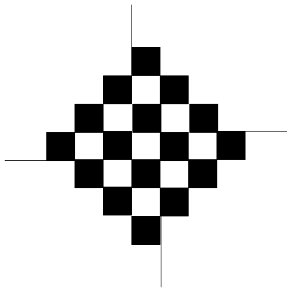


Figure 8.3 A 7-pattern rotated 90°

The lines in this figure indicate a decomposition of the pattern into triangles. In this case the number of squares in one of the triangles is identical with T_3, where T_3 symbolise the third triangular number. In a *t*-pattern the number of squares in one of the triangles, equals the triangular number with index identical with the number of white squares at the diagonal, as mentioned at page 145 this number is $n-1$. The number of squares can therefore be calculated with the help of the formula:

$$A(t) = A(2n-1) = 4 \cdot T_{n-1} + 1 \qquad (8.8)$$

which can be simplified to:

$$4 \cdot T_{n-1} + 1 = 4\frac{(n-1)n}{2} + 1 = n^2 + (n-1)^2$$

8.3.4 Geometrical – Sum of rows or columns

Figure 8.1 can also be read as consisting of 7 rows or columns of black and white squares. Correspondingly a *t*-pattern consists of *t* rows or columns of black and white squares. The number of squares in a *t*-pattern can then be calculated as:

$$\begin{aligned} A(t) &= 1+3+5+\ldots+(t-2)+t+(t-2)+\ldots+5+3+1 \\ &= 2(1+3+5+\ldots+(t-2))+t \end{aligned} \tag{8.9}$$

The sum in the parenthesis is the sum of the first $n-1=\frac{t-1}{2}$ odd numbers. It is well known that this sum equals the square of the number of numbers in the sum; i.e. $\left(\frac{t-1}{2}\right)^2$. This implies that:

$$A(t) = 2\left(\frac{t-1}{2}\right)^2 + t \quad \left(= \frac{(t-1)^2}{2} + t = \frac{t^2+1}{2}\right) \tag{8.10}$$

8.3.5 Other relationships

According to Gerdes (1988, 1995a) the pattern investigated in this task has a long tradition as a decorative motif all over the world. This is particular the case for the whole of Africa where the pattern can trace its roots back to ancient Egypt (Gerdes, 1993, 1994). Gerdes has explored the pattern and he has demonstrated that it can be transformed into a square. As a consequence of this transformation he points out that the pupils can use this pattern as a basis for conjecturing the *Pythagorean Theorem*. He has also demonstrated that the pattern can be used to construct proofs for both the general *Pythagorean Theorem* and for *Pappos's Theorem* (Gerdes, 1988, 1993, 1994, 1995a, 1995b). In addition Gerdes (1995b) mentioned that the pattern or parts of it can be used to conjecture some number relationships. For example:

The two well-known sums:

i) $1+2+3+\ldots+(n-1) = \frac{n^2-n}{2}$;

ii) $1+3+5+\ldots+(2n-1) = n^2$;

and the more spectacular:

iii) $$\begin{aligned} &n\cdot 1+(n-1)\cdot 2+(n-2)\cdot 3+\ldots+3\cdot(n-2)+2\cdot(n-1)+1\cdot n \\ &= \frac{1^2+2^2+3^2+\ldots+(n+1)^2-(1+2+3+\ldots+(n+1))}{2} \end{aligned}$$

....

8.4 Class M

Attached to this answer was a description from the teacher where he gave a survey over the method used by the class. He wrote:

> Firstly I let the pupils draw figures with different sizes, colour the squares black and white, and find the number of squares across. Then I draw on the overhead projector, while the pupils were looking, figure 1 with 3 squares across.
> The pupils then got the task of finding the number of squares across, the number of black squares, and the number of white squares. This was written in a table on a transparency.
>
> | Across. | Black. | White. |
> |---|---|---|
> | 3 | 4 | 1 |
> | 5 | 9 | 4 |
>
> We expanded with one square in length and width, and found the number of squares across, the number of black ones and white ones. (…)
> We continued like that until we had a figure with 13 squares across.
> The pupils then got the task of exploring the table which we had constructed, and find relationships between the numbers in the table. These relationships were written down as the statements 1-9. (Class M, p.1-3)

It is noticed that the teacher designed the table and then guided the class to explore the numbers that was put in the table. The nine statements were the following:

1. The four blacks with the three across fitted in into the four whites with the five across.
2. The nine blacks with the five across fits into the nine white with the 7 across etc.
3. On the blacks we have four and when we move over to the whites and moves one step down we find the same number. And that continues and continues downwards.
4. The blacks minus the whites equals the across.
5. The across plus the whites equals the blacks.
6. The blacks minus the across equals the whites.
7. The across is it only odd numbers.
8. The across increase with two at the time.
9. The blacks are even numbers and odd numbers all the time downwards and on the whites it is in reversed order. (Class M, p. 4-5)

Some of these nine statements focus on the same relationship, and they can be classified with respect to three main groups.

- Relationships between two neighbouring patterns: Statements 1-3;
- Relationships within one pattern: Statements 4-6;
- Relationships linked to parity: Statements 7-9.

Using the symbols and notations from page 144, these statements can be decoded into a mathematical language and be written:
Relationships between two neighbouring patterns:

1. $B(3) = W(5)$;
2. $B(5) = W(7)$;
3. $B(3) = W(5)$;
   $B(5) = W(7)$;
   $B(7) = W(9)$;
   $\vdots$

Relationships within one pattern:

4. $B(t) - W(t) = t$;
5. $t + W(t) = B(t)$;
6. $B(t) - t = W(t)$.

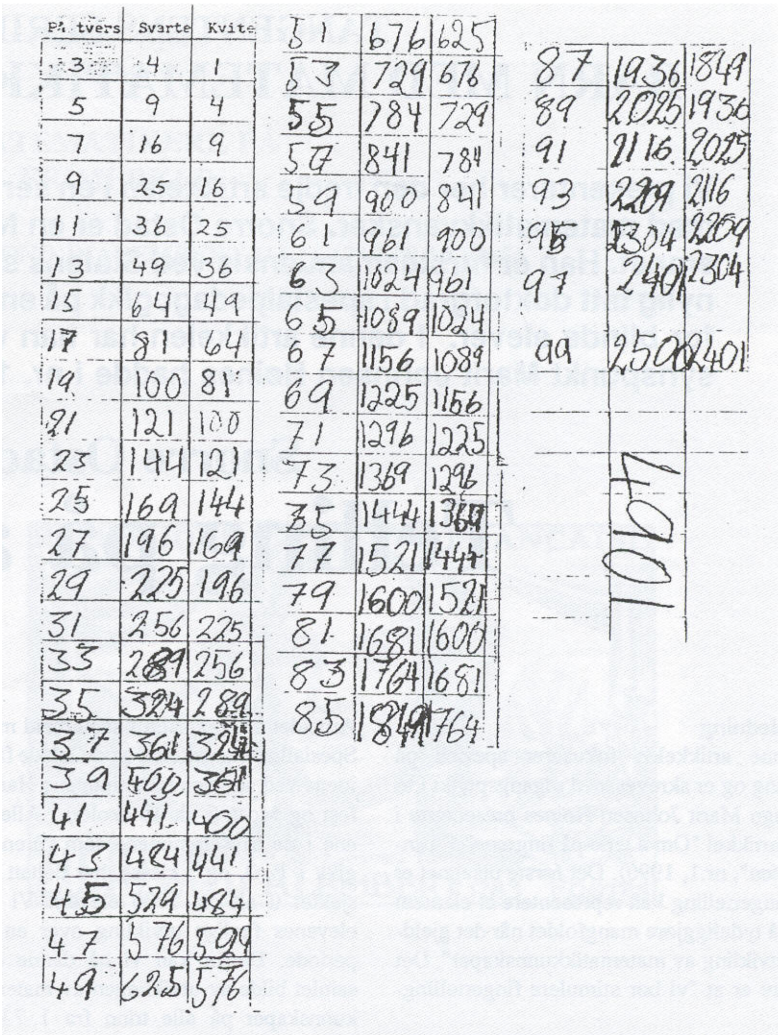
Relationships linked to parity:

7. t is an odd number;
8. the t-values increases by 2;
3. $B(t)$ and $W(5)$ have opposite parity.

One of the challenges for the pupils was to determine the number of squares in the 99-figure. With these nine statements as a basis the teacher challenged his class:

> Can we, without making more drawings, use what you have discovered go on and complete the table?
> That they quickly managed, (…) with four pupils in each group they started to calculate down to 99 squares across (…) (Class M, p.1)

All together five groups of pupils worked with these calculations. The result of one of these groups is shown in figure 8.4. All the completed tables are found at the pages 7 and 8 in the solution to this class. Class M formulated the final result in statement number ten, which concluded that there all together were 4901 squares in the 99-figure. This is correct, see page 145.



På tvers	Svarte	Kvite
3	4	1
5	9	4
7	16	9
9	25	16
11	36	25
13	49	36
15	64	49
17	81	64
19	100	81
21	121	100
23	144	121
25	169	144
27	196	169
29	225	196
31	256	225
33	289	256
35	324	289
37	361	324
39	400	361
41	441	400
43	484	441
45	529	484
47	576	529
49	625	576
51	676	625
53	729	676
55	784	729
57	841	784
59	900	841
61	961	900
63	1024	961
65	1089	1024
67	1156	1089
69	1225	1156
71	1296	1225
73	1369	1296
75	1444	1369
77	1521	1444
79	1600	1521
81	1681	1600
83	1764	1681
85	1849	1764
87	1936	1849
89	2025	1936
91	2116	2025
93	2209	2116
95	2304	2209
97	2401	2304
99	2500	2401

Figure 8.4 The completed table to one of the four groups in Class M (*Tangenten* **2**(4), p. 26)

In the quotation above the teacher informs that the class, with the nine statements as a basis, quickly discovered how they could calculate in order to determine the number of squares in the 99-pattern. The class had filled in the table to $t = 13$, and know the values of $W(13)$ and $B(13)$, and then also $A(13) = W(13) + B(13)$. The goal was $A(99) = W(99) + B(99)$.

From statement number 8, they know that the next *t*-value was $t = 15$. From the statements 1-3, which linked neighbouring patterns, they could calculate $W(15)$. The statements 4-6, which gave relationship within one pattern, the value of $B(15)$ could then be calculated. The calculation of $A(15)$ was then just an addition. All together this meant that the next line in the table had been filled in. Repeating this process would then give the next line in the table, and another repetition the next etc.. Mathematically, this can be characterised as an induction process which make use of two different recursive relations and one explicit formula. A description of this process, using mathematical symbols, can be:

The starting values are known, here $t = 13, W(13)$ and $B(13)$.

For $t \geq 13$ the relationships between the three different values are given by:

1. next $t = t + 2$
2. $W(t+2) = B(t)$
3. $B(t+2) = W(t+2) + (t+2)$

This solution is identical with the one presented as (8.3) in section 8.3.1. The relationships 1 and 2 above are the recursive formulas, and relationship 3 is the explicit formula.

The tables completed by Class M, exemplified by figure 8.4, could also be regarded as function tables, with *t* as the independent variable, and $W(t)$ and $B(t)$ as two different dependent variables.

8.4.1 Summary – Class M

In this task the pupils were encouraged to investigate different sizes of the pattern in question. The numbers obtained by this investigation process were put in a table, which the teacher had designed. The pupils then examined or studied these numbers and several relationships were discovered and written, not all the relationships written were different. These relationships can be grouped in three different groups; relationships between two neighbouring patterns, relationships within one pattern, and relationships related to parity. Based on three of these relationships it was then possible to calculate the number of squares in the 99-figure. Two of the relationships used in this calculation were recursive formulas and one was relationship was given as an explicit formula. The strategy used in this calculation can be characterised as an induction process. The solving procedure rested heavily on two recursive formulas and one explicit formula. Functions are also inherent in this solution procedure.

Mathematical structures inherent in the solution procedure to Class M:

- Recursive defined functions;
- Explicit defined functions;
- Induction principle.

8.5 Class N

This was the second time this class participated in the *Tangenten* competition, their first participation was with a solution to the task *A box with ten cubes*; in this study called Class D. One of the challenges addressed in this task was to solve the problem in different ways. Class N prepared two different solutions, in this section named solution A, and solution B. Attached to the solution were four big cardboard on which Class N had constructed the seven smallest patterns; i.e. from the 1-pattern to the 7-pattern inclusive. The size of each of the black and white squares was $5 \times 5 \text{ cm}^2$.

8.5.1 Solution A

At the first page in their solution Class N presented the following table:

No. 1	$1 \cdot 4 + 1$	=	5
No. 2	$3 \cdot 4 + 1$	=	13
No. 3	$6 \cdot 4 + 1$	=	25
No. 4	$10 \cdot 4 + 1$	=	41
No. 5	$15 \cdot 4 + 1$	=	61
No. 6	$21 \cdot 4 + 1$	=	85

Figure 8.5 The solution A to Class N

Figure 8.5 reveals that Class N perceived the *t*-patterns as made up of four triangles and a single square. The calculations have been carried out with a formula identical with (8.8). This solution is therefore identical with the solution presented in section 8.3.3. The table in figure 8.5 may be regarded to consist of the following four columns:

m	Triangular number m	The expression $T_m \cdot 4 + 1$	Total number of squares

The two last columns in the table gives the same information, and can be understood to be identical. As can be seen from figure 8.5 Class N did not operate with a separate column for the triangular numbers. They have just put these numbers into the expression $T_m \cdot 4 + 1$, and then completed the calculations. The explanation for doing it like that is later on in their text brought to the light when they wrote: “We have the triangular numbers on the wall in the classroom.” (Class N, p.1). There was therefore no need for a separate calculation of the triangular numbers.

The table in figure 8.5 may also be interpreted as a function table for a composite function. The independent variable is the *m*-values the dependent variables are the triangular numbers, and the expression $T_m \cdot 4 + 1$. This can by illustrated by the figure:

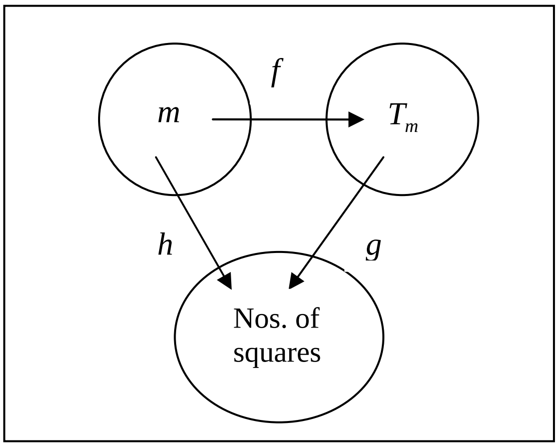


Figure 8.6 Illustration of the functions found in table in figure 8.5

Explanation of the symbols in figure 8.6:

m is the independent variable, $m \in \{1,2,3,4,5,6\}$, and
f is the function defined by $f(m) = T_m$, the T_m-value is found 'on the wall', and
g is the function defined by $g(T_m) = T_m \cdot 4 + 1$, and
h is the function defined by $h(m) = T_m \cdot 4 + 1$.
This means that $h(m) = g(f(m))$ or $h = f \circ g$.

The relationships between m, n, t and $A(t)$ are given by:

$$m = n - 1, \quad t = 2m + 1 \text{ and } A(t) = A(2m+1) = T_m \cdot 4 + 1.$$

It is noticed that Class N did not state a value for $A(99)$. The main reason could be that they did not have the value to T_{49} 'on the wall', or they did not know the formula for T_n, and was therefore unable to do the necessary calculations. Another reason could bee that they not had observed or been able to figure out the relationship between *m* and *t*, i.e. that $t = 99$ corresponded with $m = 49$, and was therefore not able to put up the expression $A(99) = T_{49} \cdot 4 + 1$. Looking at their solution B it seems most likely that they only know the T_n-values for small values of *n*.

8.5.2 Solution B

At page 3 Class N presented the following table:

	Black		White		
No. 1	1·1	+	0·0	=	1
No. 2	2·2	+	1·1	=	5
No. 3	3·3	+	2·2	=	13
No. 4	4·4	+	3·3	=	25
No. 5	5·5	+	4·4	=	41
No. 6	6·6	+	5·5	=	61
No. 7	7·7	+	6·6	=	85
No. 99	99·99	+	98·98	=	19405

Figure 8.7 Solution B to Class N

Figure 8.7 reveals that Class N here has decomposed the pattern into a sum of two quadratic patterns, and calculated the number of squares by a formula analogous to (8.5). The numbering used in this table is in accordance with the numbering on the four big cardboard. Based on figure 8.7 one can conclude that the class realised that the number of white squares at the side in the white quadratic pattern in one *t*-figure, is one less than the numbers of black squares in the black quadratic pattern in the same *t*-figure.

Figure 8.7 reveals also that Class N in this solution has interpreted a 99-pattern to have 99 black squares across, and not a total of 99 black and white squares. By reason of the discrepancy between the text of the task and the pattern given in the task, see page 143, this is an acceptable interpretation. This interpretation made the determination of the total number of squares in a pattern a bit simpler than in the case of a main diagonal of length 99 squares. If the length of the main diagonal is given, it is necessary to determine the number of both black and white squares on the respective quadratic patterns; i.e. using the numbering to Class N to realise that the figure in question is not no. 99, but no. 50. It is noticed that Class N has not a consistent interpretation of the phrase ‘squares across’. At page 2 they used this phrase for the main diagonal, which consists of both black and white squares.

The table in figure 8.7 can also be considered as a function table, where the left column represents the independent variable, i.e. the number of (black) squares at the side of the black quadratic pattern. The right column represents the dependant variable, the total number of square in the pattern. If this independent variable is called *r*, then Class N has used the formula

$$A(r) = r \cdot r + (r-1) \cdot (r-1)$$

in their calculations. As shown in section 8.3.2, this is a correct formula for calculating the total numbers of squares in a pattern.

8.5.3 Summary – Class N

The structure of this task guided the pupils to investigate patterns of different sizes, and determine the total number of squares in the patterns. The pupils were also encouraged to find more that one way or method to carry through their investigations. In their solution Class N has attacked the problem via two different geometrical decompositions. Both methods resulted in correct explicit expressions. For both solutions it is argued that Class N, mathematically, has made use of functions in one variable, and with regard to the solution A, of a composite function. Triangular numbers plays an important role in the solution A.

Mathematical structures inherent in the solution procedures:

- Explicit defined functions;
- Composition of functions;
- Figurative numbers (Triangular numbers).

8.6 Class O

As was the case for Class N, this class presented also two different solutions. Both solutions relied on geometrical decomposing of the pattern. Solution 1 applied the decomposing described in section 8.3.2 and solution 2 on that from section 8.3.4.

8.6.1 Solution 1

This solution is fairly identical with the solution B to Class N, see section 8.5.2 page 153. In order to demonstrate the relationships the pupils made use of illustrations, and calculated the number of squares found in these illustrations. In the solution it was exemplified with the 3-, 5- and 7-figures. At one point this answer differs from the answer to Class N. Class O interpreted the task as was the intention; i.e. that 99 was the number of squares at the main diagonal, and not the number of black squares at the side of the black quadratic pattern. This appears from what they wrote at page 3:

> We find out how many black squares there are vertically downward and multiply this with oneself. We do the same with the white squares.
> If we had 99 squares across, then it becomes 50 black vertically and 49 white vertically, and we can calculate like this

	$50 \cdot 50$	black	=	2500	black
+	$49 \cdot 49$	white	=	2401	white
			=	4901	black and white squares

(Class O, p. 3)

The description given in the two first lines in the quotation above is not restricted to a particular *t*-figure; it has a general form. Even though this description, from a mathematical perspective is a bit defective, the context and the former examples given by the class, the decoding causes no difficulties. The interpretation has to be:
The total number of black squares in an arbitrary *t*-figure is given as:

(the number of vertical black squares in the pattern)^2,

and correspondingly for the total number of white squares in such a figure is given as:

(the number of vertical white squares in the pattern)^2.

Then the total number of squares is the sum of those two numbers. The problem of finding the total number of squares in a *t*-figure was then reduced to find the number of black vertical squares, and the corresponding number of white squares.

In their answer Class O clarified the distribution of the black and white squares at the main diagonal; the number of black squares on the main diagonal was one more than the number of white squares at the same diagonal. And more than that, these two numbers were identical with what Class O called the number of vertically (squares). The number of black squares at the main diagonal was identical with the number of black squares at the side of the black pattern, and correspondingly for the white squares. It appears not from their answer how the number of black respectively white squares at the diagonal was calculated; i.e. 50 and 49. If the number of black squares at the main diagonal is called *n*, there is no indication that they applied the relationship $n = \frac{t+1}{2}$. The relationship conjectured by Class O can be written as: $A(t) = n^2 + (n-1)^2$, which is identical with (8.7). In a mathematical language this is an explicit function in one variable.

8.6.2 Solution 2

Concerning their second solution Class O wrote:

> We found out how many squares it was across in the middle and on the sides
> $5+2\cdot 3+2\cdot 1=13$
>
> $7+2\cdot 5+2\cdot 3+2\cdot 1=25$
> Here we se that all the numbers we multiply by 2 is odd numbers in series from 1, the last element is the number of squares across. (Class O, p. 4)

This is a very good description of the figures considered, and it is quite simple to write down the sums for the small figures. In the two presented examples the number of elements in the sums is small, and it is therefore relatively simple to calculate these sums. For big numbers the problem is more complex, the sum contains many elements, and accordingly many additions. On the basis of the description the answer shows that Class O did realise that the corresponding sum linked to the 99-figure had to be:

$$99+2\cdot 97+2\cdot 95+\ldots+2\cdot 5+2\cdot 3+2\cdot 1 \tag{8.11}$$

which is formula (8.9) with $t=99$.

The challenge to Class O was then: How to determine this number without doing all the additions? Which can be reformulated to: Find a formula for calculating the sum given in (8.11). In their attack on this question Class O used the following strategy. They had, as seen from the quotation above, observed that except for one element in the sum (8.11), the elements were odd numbers in series that should be multiplied by 2. The problem was then transformed to calculate $99+2\cdot(97+95+\ldots+5+3+1)$. Since they did not have a formula for calculating series like $97+95+\ldots+5+3+1$, they started to investigate sums of odd numbers in series starting with 1. In their answer is found the first four smallest of these sums:

> $1+3=4=2\cdot 2$
>
> $1+3+5=9=3\cdot 3$
>
> $1+3+5+7=16=4\cdot 4$
>
> $1+3+5+7+9=25=5\cdot 5$
> (Class O, p. 4)

It is not known if they investigated more than these four sums, in any case they concluded "We find the sum of odd numbers in series from 1 and up by multiply the number of elements by itself." (Class O, p. 4). As is well known, it is a correct observation that the sum of odd numbers in series, starting with one, equals the square of the number of elements in the sum. Class O did not give any further argument for the correctness of their conclusion. The next challenge was then to determine the number of elements in the series $97+95+\ldots+5+3+1$. Class O stated that this number was 49, which is correct. From their answer it is not clear how this number was fixed. This conclusion could have been based on several possible strategies:

1. Count all the odd numbers from 1 to 97.

2. Less than 100 there are 50 odd numbers, accordingly there are 49 odd numbers to 97 inclusive.
3. They could have tried, and checked against the result known from their former solution.
4. Observed from the four written sums, that the number of elements in the sums was given as the biggest number plus 1 divided by 2, and therefore for this sum the number elements was $\frac{97+1}{2} = 49$.

In retrospective it is impossible to state if Class O really used one of these strategies. In this context this is not seen as an important question. In a way is it more interesting why they did not find it necessary to argue for the correctness of the number 49. The answer to this last question could be so simple that they did not se any reason to do it, it was obvious that the number had to be 49. In any case they concluded:

> If we have 99 squares across, we can calculate:
> $99 + 2 \cdot 49 \cdot 49 = 4901$
> This number is identical with that from the former solution. Hurrah! (Class O, p. 4)

The phrase 'If we have 99 squares across' is a strong indication that applying this method Class O could have calculated the number of squares an arbitrary *t*-figure. Their problem was now reduced to determine the number of elements in a series, and Class O would most likely be able to find that number. Mathematically, determining that number is to handle a function in one variable. If the symbol *N* is used for that function then

$$N(n) = \begin{cases} \text{The number of elements in the series } 1,3,5,\ldots n, \\ \text{where } n \text{ is an odd number} \end{cases} = \left(\frac{n+1}{2}\right)^2$$

thus their second method can be symbolised:

$$A(t) = t + 2N(t-2)^2 = t + 2\left(\frac{t-1}{2}\right)^2 \quad \left(= \frac{t^2+1}{2}\right) \qquad (8.12)$$

which tells that composition of functions is involved. As known from section 8.3.4 (8.12) is correct, and it is identical with (8.10) page 147. In this solution Class O used odd numbers and square numbers; i.e. figurative numbers. Their relationship can be decoded into a second-degree explicit expression in one variable.

8.6.3 Summary – Class O

Class O presented two different solutions. The solutions were based on two different geometrical decompositions of the pattern. In the first solution they used the method described in section 8.3.2, sum of two squares. The second solution utilised that the pattern can be decomposed into rows or columns, as demonstrated in section 8.3.4. Even though both solutions focused on the number of squares in the 99-figure it is argued that Class O would have been able to calculate the number of squares in an arbitrary *t*-figure. Both methods resulted in two correct explicit expressions in one independent variable. Mathematically they used functions in one variable.

In the second solution Class O investigated sums of odd numbers in series and they discovered the relationship between such sums and square numbers. This relationship was stated in a general version. In this investigation figurative numbers played a decisive role.

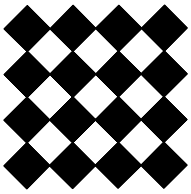
Mathematical structures inherent in the solution procedure:

- Explicit defined functions;
- Composition of functions;
- Figurative numbers.

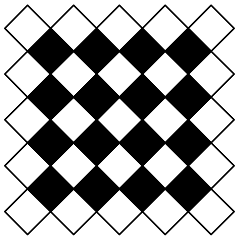
8.7 Class P

A letter from the teacher (Class P, p. 1), attached to this solution, informed to a certain extend how Class P had been organised when they worked on this task. The number of pupils in Class P was seven, which means that may be called a ‘small class’. They started to work on this particular task after the end-of-the year examination in June. The working period was organised in two periods. In the first period the class worked collectively, in the second period they were organised in three groups. Based on the explorations from the first working period, each of two groups, both groups had two members, used *WordPerfect* and prepared a draft solution. One of these two drafts was then revised in order to cover all the discovered relationships. This revised solution was then used in the competition. The third group, which had three members, constructed in *PlanPerfect*, a spreadsheet-table that should explain the system or how the numbers of squares in the figures grow when the number of squares at the diagonal increased. There was however no particular information about the way Class P worked on the task. Most likely, in the first period, they worked in the way indicated in the task; i.e. they investigated the figures for small values of *t*. On the basis of these investigations the different relationships were formulated.

This answer unveils that, compared with the other classes; Class P had an alternative interpretation of the task. This appears especially from the spreadsheet-table (Class P, p. 3), but also from what they wrote at page 2 “- If a vertical line starts with black it ends in black. The same is true for white.” From the constructed spreadsheet table (Class P, p. 3) it appears that the 1-figure consisted of one white square. Most likely this square can be identified as the white square in the middle of the 7-figure which illustrated the pattern focused on in this task. For the next figure, the 3-figure, the rows and columns started and ended in black squares. Continuing to bigger figures the colour of the starting and ending squares in the rows and columns alternate between white and black. For example the rows and columns in the 7-figure starts and ends with a black square, and in the 9-figure the rows and columns starts and ends with white squares, see the figures below:



The 7-figure



The 9-figure

Figure 8.8 Example of how Class P interpreted the patterns

However, this interpretation does not have any influence on the total number of squares in a pattern. In a *t*-figure the number of squares, $A(t)$, is the same irrespective of a row or a column starts and ends with white squares or black squares, it is only the reciprocal distribution between the two colours that changes. If $t \equiv 1 (\mathrm{mod}\, 4)$ this interpretation implies that the rows and the columns in a *t*-figure starts and ends with white squares. In such a figure the distribution between the black and the white squares is reversed compared with a *t*-figure where all the rows and columns starts and ends in black squares. If $t \equiv 3 (\mathrm{mod}\, 4)$ the rows and columns in the *t*-figure starts and ends with black squares.

8.7.1 The WordPerfect part (solution)

As stressed elsewhere one of the challenges in this task was to look for more than one way to determine the number of squares in the 99-figure another challenge was to look for other relationships hidden in the pattern (than the number of squares in the 99-figure). Class P did both. In addition to determine the number of squares they listed altogether eleven statements concerning the pattern. Class P did not number these eleven statements, but in order to refer to them in this study they have been numbered i) to xi). These statements, found at page 2 in their solution, were organised in four sections that can be characterised:

- black and white squares, i) – iii);
- odd numbers, iv) – vii);
- answer to the task, viii) – ix);
- relationships, x) – xi).

The first seven and the ninth of these statements are listed below. The three remaining statements, viii), x) and xi), will be dealt with later on. Statement viii) establish the number of squares in the 99-figure and is commented on in section *Solution 99-figure* page 161. The two remaining statements, x) and xi), were written under a section Class P called relationships, and they are dealt with in the section called *Relationships*, page 163.

i) It is always one more of one colour than of the other at the diagonal.
ii) If a vertical line starts with black it ends with black. The same is true for white.
iii) If you count one colour at the diagonal and multiply it with oneself, you find out how many there are of this colour in the square.
iv) It is not possible to have an even number at the diagonal.
v) The white is ALWAYS an odd number, and the black is ALWAYS an even number.
vi) The blacks and the whites together are ALWAYS an odd number.
vii) Ergo: when you add an odd number and an even number you get ALWAYS an odd number.
ix) So many blacks as it are in the diagonal so many blacks are it vertically and horizontally. The same is valid for the white.

(Class P, p. 2)

In their solution Class P used capital letters for the corresponding Norwegian word for always.
It is noticed that except for statement vii) all the remaining statements are related to arbitrary *t*-figures, while statement vii) is a general statement about adding numbers of opposite parity.

As seen from the quotation above these relationships are stated in a condensed (written) language. Isolated from this particular context, most of these statements would be impossible or at least very difficult to interpret. However, in this context the interpretation of phrases like ‘one more of one colour’, ‘the white is’, and ‘starts with black and ends with black’ etc. causes no difficulties. It is tacitly understood that the colours refer to the colours in a pattern and the number of squares of the indicated colour in that pattern.

From the quotation above it appears also that some of the statements are related to the same quantity. For example i) and iv) deal with the diagonal. For the two statements v) and vi) it is a bit more uncertain what quantity Class P had in their minds, it could either be the number of squares at the diagonal or the number of squares in a figure. These two statements are correct for both quantities. However, it is found most likely that both of these statements, v) and vi), were related to the diagonal. The main reason for this is the context. These statements follows directly after a statement, iv), which dealt with the diagonal, and that all these three statements, iv), v) and vi), were located in the ‘odd numbers’ section, see page 159. Some of the statements overlaps or can be deduced from each other, for example statement iv) is a consequence of statement i). This follows from the fact that statement i) implies that the number of black squares and the number of white squares at the diagonal are neighbouring numbers, which mean that one of these two numbers had to be an odd number, and the other number an even number. The well known fact that the sum of two numbers of opposite parity is an odd number is statement vii). Since the diagonal was the sum of two numbers of opposite parity, the number of squares at the diagonal had to be an odd number; i.e. statement iv). Statement iii) ascertain that the number of black squares in a figure is a square number, and correspondingly for the white squares. It appears as the statements v) and

vi) are used as arguments for the correctness of statement vii). Mathematically statement ix) establish that there is a one-one correspondence between the black squares at the diagonal and the black squares at the vertical ‘edge’ of the black squared pattern, and correspondingly for the white squares at the diagonal. As known from section 8.3.2 all these statements, i) to vii) and ix), are correct. Mathematical structures inherent in these statements are figurative numbers; odd/even and square numbers.

Solution 99-figure

Statement viii) is the only one of the eleven statements that was directly related to the 99-figure. This statement answers the question of the number of squares in the 99-figure, and it was formulated:

> It is 4901 squares in a pattern that has 99 squares in the diagonal. We found it out in this way:
>
> | | | |
> |---|---|---|
> | | 49 * 49 = 2401 | White squares |
> | + | 50 * 50 = 2500 | Black squares |
> | | 4901 | Squares all together |
>
> (Class P, p. 2)

As known form section 8.3.2 this is correct. This statement reveals that Class P solved this task by using the method called sum of two squares, described in section 8.3.2. The underlying argument for this method is found in statement iii):

> If you count one colour in the diagonal and multiply it with oneself, you find out how many there are of this colour in the square. (Class P, p. 4)

The solving of the problem was then reduced or transformed to a counting and a squaring procedure. The purpose of the counting procedure was to determine the number of black and white squares on the diagonal. These numbers can be fixed with a direct counting procedure executed on a 99-figure; the black and white squares are physically counted, but they can also be fixed by an indirect counting procedure; a quantity in the figure is used as a counting object or medium. The question is: How did they pick up that the numbers 49 and 50 were the numbers of respective white and black squares at the diagonal? It is found very likely that Class P applied an indirect counting procedure on the 99-figure, the main reason for this statement is the size of the figure, and that an indirect procedure could be performed relatively easily. They know that the sum of the two numbers in question was 99. From statement i) they were aware that one of the numbers was one more than the other; i.e. that they were neighbouring numbers. This implied that the two numbers had to be 49 and 50, but remembering their interpretation of the task, see page 158, the question was; which colour was related to the number 49 or alternatively to the number 50? Statement v) established a relationship between the two colours and the parity of the number of black and white squares. At page 160 it was argued that statement v) most likely was related to the number of squares at the diagonal. In that case it was straightforward to ascertain that the number of white squares at the diagonal had to be 49, since this was the odd number. Accordingly the number of black squares had to be 50. If the statement v) not was related to the diagonal, but to the number of squares in a figure they still know that the number of squares for the two colours was 49*49 and 50*50. Of these two numbers the first is an odd number and the second is an equal number, which mean that the number of white squares is 49*49.

However, for the total number of squares in a *t*-figure, it is of no importance to know if it is the number of white or black squares at the diagonal that is the odd number. For one of the two colours the number of squares at the diagonal will be $\frac{t+1}{2}$, for the other colour that number will be $\frac{t-1}{2}$. Class P did not in their solution indicate the explicit relationships $\frac{t+1}{2}$ and $\frac{t-1}{2}$ for the number of squares at the diagonal, but that does not mean that they were unaware of these relationships. Statement i) give a strategy for finding the biggest number of squares for one of the two colours in a *t*-figure; this number is given as the *number* that satisfies the relationship $t = number + (number - 1)$. There is a short step from this relationship to the relationships that explicitly give the two numbers of coloured squares.

Statement iii) establish that Class P was aware of the fact that the corresponding numbers of coloured squares of each colour in a *t*-figure, was the square of those numbers; i.e. $number^2$ and $(number - 1)^2$. It is therefore reasonable to suppose that even though statement viii) is a statement about the 99-figure, Class P was aware that this method could be used to calculate the total number of squares in an arbitrary *t*-figure. This is also supported by the way the spreadsheet table was constructed, see section 8.7.2. The procedure which it is argued that Class P used can be symbolised by $A(t) = number^2 + (number - 1)^2$ which is a second-degree function in one independent variable.

The task challenged also the pupils to look for more than one way to determine the number of squares in the 99-figure, and Class P explained a second method for calculating this number by the statement x).

> The diagonal multiplied with the number of the colour it is most of in the diagonal, minus the number of the colour it is least of in the diagonal, is all the squares together.
>
> (Class P, p. 2)

This statement has a general character and establishes a relationship for an arbitrary *t*-figure. It is not particular related to the 99-figure. The above relationship is explained very well, and the interpretation causes no difficulties, the decoding into a mathematical language is therefore unambiguous. The phrase ‘the number of the colour’ refers to the number of squares with this colour. As pointed out earlier, in a *t*-figure the number of squares at the diagonal in the colour which appears most frequently is $\frac{t+1}{2}$. The corresponding number of squares for the other colour is $\frac{t-1}{2}$. In a symbolised mathematical language relationship x) can be written as:

$$A(t) = t \cdot \frac{t+1}{2} - \frac{t-1}{2} \quad (8.13)$$

A geometrical interpretation of relationship (8.13) is that the pattern can be transformed to a rectangle with base *t* and height $\frac{t+1}{2}$ minus a 'line' of length $\frac{t-1}{2}$, see the figure below:

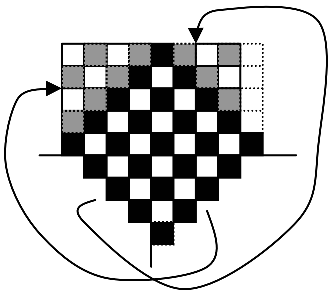


Figure 8.9 Transformation of the pattern to 'nearly' a rectangle

There is no indication that Class P realised this geometrical interpretation of the relationship. This expression can be transformed to $\frac{t^2+1}{2}$ which is another formula for $A(t)$, see section 8.3.2. The relationship x) is a correct explicit expression. It is a second-degree function in one independent variable.

Relationships

As mentioned at page 159 this answer contained also a section called relationships. This section contained two statements, which in this study is numbered x) and xi). Statement x) has been discussed in the preceding section. Class P formulated statement xi) as:

> Look at the answer of the first task. It is 99 in the diagonal. The difference between 2500 and 2401 is 99. The same is valid for other numbers. Example:
> 17 in the diagonal: $8*8=64$ $9*9=81$ $81-64=17$ (Class P, p. 4)

Even though Class P did not explicitly claim that statement xi) was valid for all *t*-figures, it is found most likely that the interpretation has to be; that this statement is valid for an arbitrary *t*-figure, where it is tacitly understood that *t* had to be an odd number. The main reason for this interpretation is the sentence "The same is valid for other numbers." (Class P, p. 2). In a strict mathematical language the term 'other numbers' does not necessarily mean *all* (*odd*) *numbers*, it could mean *some* (*odd*) *numbers*. To expect that a class of grade 7 should be aware of this distinction is found to be of little realism, so most likely it had to be tacitly understood that the statement is valid for all (odd) numbers.

As seen from the quotation above the statement claims that the difference between two numbers equals a third. The number of squares at the diagonal is identical with a difference between two numbers, but it does not explicitly state which two numbers to be subtracted. From the given number examples it appears that these two numbers are the total number of squares of each colours in the figure; i.e. the square of the number of black squares at the diagonal, and correspondingly for the white squares. The number of black squares at the diagonal is, as remarked elsewhere, one of the numbers $\frac{t+1}{2}$ or $\frac{t-1}{2}$, and the remaining number is the corresponding number of white squares. In a symbolised mathematical language this relationship can therefore be formulated as $\left(\frac{t+1}{2}\right)^2 - \left(\frac{t-1}{2}\right)^2 = t$. Again Class P stated a correct relationship. Related to the interpretation Class P had to this task, the lines and rows in a figure could also start and end with white squares, this relationship is equivalent with the relationship (8.2), see section 8.3.1.

8.7.2 The PlanPerfect part (solution)

One group of three pupils prepared, as mentioned at page 158, a spreadsheet-table in order to show or illustrate the discovered relationships or, as their teacher put it the attached letter “the development of the growing size of the squares” (Class P, 1991). The spreadsheet table range from $t = 1$ to 113. An natural question is therefore: How did Class P prepare their spreadsheet? or formulated as: Which formulas or instructions were programmed in the different cells in the spreadsheet? One possibility was to calculate each of the actual numbers manually, and then write the calculated numbers in their respective cells in the spreadsheet. That could be characterised as using the spreadsheet as a manual table. In that case the strength of the spreadsheet had not been utilised. It is found very unlikely that Class P used the spreadsheet like that, most likely the class constructed formulas that were copied to other cells in the spreadsheet. This view is also supported in the letter from the teacher:

> We had not used PL that much so I had to help them with instructions for copying and formatting of the cells. The formulas they constructed themselves. (Class P, p. 1)

One strength or advantage to a spreadsheet is the way the values of different cells can be combined in order to create values to other cells. To be more precise, a value in one cell can be constructed or calculated on the basis of earlier given values. This construction or calculation can be based on two different principles; a calculation can be performed either with a recursive-defined formula or with an explicit-defined formula.

To answer the question, which formulas were programmed in the cells, on the basis of a printed table is, however, an arduous task. In order to facilitate and also ensure that a reconstruction was done or could be done in accordance with the original solution, the teacher of Class P was contacted. A letter that asked for a copy of the spreadsheet file, if it still existed, was mailed. The time this letter was mailed, was however, about three years after the pupils had worked out their answer. There came no written response from the teacher. Later on, in August 1997, the teacher personally informed that he remembered this letter, but since he neither had a copy of the file, and nor remembered the pupils solution, he could not contribute with any further information, so he therefore did not respond to the letter. See also page 61.

The table constructed by the class contains eight columns. If, as usual for spreadsheet, the letters A to H are used for identifying the columns, the correspondence between the heading to each of the columns and the letters are:

A: The number of squares at the diagonal;
B: The increase in the number of squares at the diagonal;
C: The number of black squares in the figure;
D: The increase in the number of the black squares in the figure;
E: The number of white squares in the figure;
F: The increase in the number of the white squares in the figure;
G: The total number of squares in the figure;
H: The increase in the total number of squares in the figure.

The rows in a spreadsheet are usually numbered by numerals. For practical reason the first row in the table that contains only numbers will in this study be called 1, the next row for 2, etc.. This means that every number in the table is identified by a letter and a numeral, for example the number in the upper left corner is identified as the cell A1. Using this notation the construction of a spreadsheet can be explained. The question is then if it is possible, retrospectively, to reconstruct the spreadsheet applied by the class. For most of the columns this reconstruction seems to be possible, that is particular the case for the columns A, B, D, F, G and H. While for the columns C and E the reconstruction is more uncertain. Most likely the cells in the A column was defined by: $A1 = 1$ and $A2 = A1 + 2$, and the last identity was then copied to the other cells in the column. This interpretation is supported by the statement iv) which claims that the number at the diagonal had to be an odd number (see page 160). Adding 2 to an odd number generates the next odd number, and since the starting value was 1, all the odd numbers can be generated by this relationship. The values to the cells in this column have then been given by repeated use of a first order linear recursion formula; one value is dependent on the value in the cell located above and in the same column. One could characterise this as a 'vertical' calculation in the spreadsheet.

The four columns B, D, F, and H deal with 'increases' between numbers. From the given table it is realised that these 'increases' mathematically are subtractions which make it very likely that the cells in these four columns have been defined as subtractions. To be more precisely the cell B2 was defined as $B2 = A2 - A1$ (the cell B1 was not defined), then the formula for cell B2 was copied to the remaining cells in this column. The cells in the column H were defined in a similar way as the cells in column B.

The cells in the columns D and F were defined in a nearly similar way to the cells in column B; the cells D1, F1, and F2 were not defined, and the next cells in the respective columns was defined as $D2 = C2 - C1$ and $F3 = E3 - E2$. Then the pair of cells respectively D1, D2 and F2, F3 was copied to the remaining cells in their respective columns.

The straightforward definition of the G1 cell is $G1 = C1 + E1$, and it is most probable that Class P used this definition, and that this formula was copied to the other cells in the column. The main reason for this claim is found in the fact that this also was the strategy used by Class P when they calculated the number of squares in the 99-figure, statement viii).

As argued above it is most likely the values in the cells in all of these five columns, B, D, F, H and G, have then been defined by explicit formulas. The values in the cells are dependent on values found in cells located to the left of the column, which can be characterised as a ‘horizontal’ calculation in the spreadsheet.

For the two remaining columns, C and E, the reconstruction is found to be more difficult or uncertain. This uncertainty was the main reason for addressing the letter to the teacher of this class, see page 164. Since no response was received on this inquiry it is considered to be of little interest or importance to speculate how these two columns was constructed. It is, however, noticed that the cells in the C column can be constructed by the following procedure: Set $C1 = 0$, $C2 = ((A2+1)/2)*((A2+1)/2)$ $= ((A2+1)/2)\,\hat{}\,2$, and $C3 = C2$, and then copy the pair of cells C2 and C3 to the other cells in the column. Column E can be constructed in nearly the same procedure as column C. There is, however, no evidence that Class P constructed these two columns in this way.

8.7.3 Summary – Class P

This solution was organised in two parts, a WordPerfect part and a PlanPerfect part. The WordPerfect part consisted of eleven correct statements. One statement was a general statement about the adding of numbers of opposite parity, and one was linked to the 99-figure only. Each of the remaining nine statements was statements about at least one relationship concerning an arbitrary *t*-figure. Mathematical structures found in many of those statements are odd/even numbers, and square numbers, which are figurative numbers.

Class P presented two different methods for determining the number of squares in a *t*-figure. One method was given in their solution for the 99-figure, and is identical with the solution procedure called the sum of two squares, presented in section 8.3.2. It is argued that Class P could have used this method on an arbitrary *t*-figure, and that they demonstrated the correctness of the method. The other method applied a relationship based on observations. Class P observed a particular relationship between four different numbers related to a *t*-figure; the total number of squares, the number of squares at the diagonal, and the number of respectively black squares and white squares at the diagonal. This relationship was stated as valid for all *t*-figures, which is correct. Both methods lead to explicit second-degree expressions in one variable; i.e. functions in one independent variable.

Un-earthing the hidden instructions used in the PlanPerfect part was, for some of the cells, an arduous task. It was emphasised that in some cases it was uncertain which formulas Class P applied in their construction of the spreadsheet. Even though it had been interesting to know the formulas used, which also had made the analysis more comfortable and consistent, it is, however, possible to give an overall description of the mathematical process carried through in the construction of the spreadsheet.

The values in the A-column, the numbers of squares at the diagonal, are an independent variable. The values in the cells in the other columns are all dependent of the values from column A; which mean that they are defined by formulas or relationships that rests, directly or via previous calculations, on the values in column A. It is argued that the relationship defining the values in column A was a first order linear recursion formula, and the relationships that defined the values in the other columns were explicit given formulas.

Mathematical structures inherent in the solution procedures:

- Figurative numbers;
- Explicit defined functions of second degree in one variable;
- Recursive defined function.

8.8 Class Q

In line with most of the participating classes, Class Q also presented two different solutions of the task. Both solutions were based on exploring the interweaving of the two quadratic patterns or what may be more descriptive the separation of the pattern into two quadratic patterns, one consisting of the black squares and one with the white squares.

8.8.1 Solution 1

The strategy behind this solution was; determine the length of the sides to the black pattern and the white pattern, square these two numbers, and then add the squared numbers. This is the solving strategy presented in section 8.3.2. For these two lengths Class Q used the descriptions “the side of the pattern” for the black pattern, and “the side of the white pattern” for the white pattern (Class Q, p. 2). Since these two lengths are dependent of the length of the diagonal, it was therefore necessary for Class Q to establish a relationship between the length of the diagonal and the length of the sides of the two patterns, the black and the white. In their answer Class Q focused on two different methods or strategies for determining this relationship.

Method 1 – Adding

The first method used the fact that the number of the black squares at the diagonal equalised the number of the black squares at the side of the black pattern, and correspondingly for the white squares. They did not direct emphasise that the number of the white squares at the diagonal was one less than the number of the black squares, it is however reasonable to interpret that they observed this relationship. This interpretation is supported from the following:

> Since the diagonal is 99 (in the above figure it is 7) then the side is 50 (4 in the figure), and the white patterns side is 49 (3 in the figure) $4+3=7$. $50+49=99$. The side of the black pattern + the side of the white pattern = the diagonal. (Class Q, p. 2)

Further supported for this interpretation is found at page 3 where Class Q has a section called "Relationships between the diagonal and the pattern". Here they again exemplify with $t=7$, and illustrate this example with a drawing. In addition they wrote:

> If the diagonal was 83 then the side of the pattern should have 42 squares, the side of the white pattern should have 41. (Class Q, p. 3)

This method for determining the length of the sides to the two patterns can be described by: Find two neighbouring numbers that add to 99, or in the general case, if the length of the main diagonal is *t* find two neighbouring numbers that adds to *t*. For small numbers and/or numbers that have a particular position as 99, neighbour to 100, this is a method that is relatively simple, and the numbers can often be determined by mental calculation. On this basis they calculated the number of squares in the 99-figure to be $50\cdot 50+49\cdot 49=2500+2401=4901$, which is correct.

Method 2 – Dividing

The second method for calculating the length of the sides to the two quadratic patterns is found at page 4 in their answer and, in a way, it is a formalising or a continuation of the first method. Their concern was now to calculate the number of squares in a *t*-figure where *t* is a relatively big number. To be more specific what are the number of squares if $t=999$ or $t=20091747$? For these two numbers, and especially for the biggest one, their first method is bothersome. The relationship between the length of the main diagonal and the length of the black and white patterns was now, page 5, given as:

$$the\ \ side\ of\ \ the\ \ pattern=\frac{999+1}{2}=500$$

$$the\ \ side\ of\ \ the\ white\ \ pattern=\frac{999-1}{2}=499$$

On this basis the total number of squares was calculated to be $500^2+499^2=499001$, which is correct. On page 5 they calculated quite analogous and correctly *the side of the pattern* for $t=20091747$. These two number examples support, very strongly, that Class Q was able to calculate, for an arbitrary *t*-figure, the length of the sides to the two quadratic patters given the length of the diagonal. The argumentation to Class Q has a general character and can be applied on a pattern of an arbitrary size.

To summarise, it is argued that Class Q had two different methods for determining the number of squares at the sides of the two patterns, the black pattern and the white pattern. The first method was based on adding the second on dividing. When these numbers were determined the total number of squares was calculated as $(\textit{the side of the pattern})^2 + (\textit{the side of the white pattern})^2$. If the length of the diagonal is called *t*, and applying the second method, dividing, Class Q calculated explicitly these two numbers to be:

$$\textit{the side of the pattern} = \frac{t+1}{2}$$

$$\textit{the side of the white pattern} = \frac{t-1}{2}$$

These two linear expressions can be identified as two linear functions. Calling the function that determine the number of respective black and white squares at one side of the pattern for *g* and *h*, these two functions can be symbolised as $g(t) = \frac{t+1}{2}$, and $h(t) = \frac{t-1}{2} = g(t) - 1$. If the squaring and adding of these two numbers are called *f*, then the relationship between the length of the diagonal and the total number of squares, used by Class Q, can be symbolised:

$$A(t) = f(g(t)) = (g(t))^2 + (g(t) - 1)^2 = \left(\frac{t+1}{2}\right)^2 + \left(\frac{t-1}{2}\right)^2$$

This is correct and is identical with (8.5).

8.8.2 Solution 2

An alternative method for calculating the total number of squares is demonstrated at the pages 4-5 in what Class Q called "Relationship no. 3". This solution is based on the observation that the difference between the number of black squares in the pattern and the number of white squares in the pattern is identical with the number of squares at the diagonal. With the symbols from section 8.3 this can be written as $B(t) - W(t) = t$, where *t* is the number of squares at the diagonal. As known from section 8.3.1 this is a correct relationship. In their answer this relationship was exemplified with the values from the 7-figure, and they did not refer to other figures than the 7-figure. It is, however, found most likely that they also investigated figures of other sizes in order to establish this relationship. They described this alternative method by:

> If one for instance know the number of black squares and the diagonal it is easy to calculate the number of white squares. One only take the black squares (16) and subtracts the diagonal (7), the answer is then 9, which is the correct answer. (Class Q, p. 4)

Calling the number of squares at the diagonal for *t*, and using the same symbols as in section 8.3, this method can be symbolised in the following way: The number of black squares in the pattern is given by:

$$B(t) = s(g(t)) = s(\frac{t+1}{2}) = \left(\frac{t+1}{2}\right)^2$$

where *s* are the squaring function and *g* are the function defined page 169.
The next step was to determine the number of white squares in the pattern. This was done with the help of the relationship $B(t) - W(t) = t$, and can be symbolised:

$$W(t) = B(t) - t = s(g(t)) - t = s(\frac{t+1}{2}) - t = \left(\frac{t+1}{2}\right)^2 - t$$

Lastly the total number of squares was fixed as:

$$A(t) = W(t) + B(t) = 2\left(\frac{t+1}{2}\right)^2 - t$$

This formula is a variant of the formula (8.10). In addition to the number example given in the explanation of the rule this method was also illustrated by another number example. Class Q selected the number $t = 20091747$, as the number of squares at the diagonal, and carried through the calculations described above. The calculations done by Class Q, in this case, contain one interesting element. Firstly, by using the earlier mentioned formula, $\frac{20091747+1}{2}$, the number of black squares at the diagonal was correctly calculated to be 10045874. Secondly, and according to the former methods the total number of black squares was correctly set up to be the answer to the multiplication $10045874 \cdot 10045874$. This was calculated to be 100919584400000 (Class Q, p. 5). The units digit, which is zero, tells, however, that this is not a correct result. Most likely Class Q carried through the multiplication on a (handheld) calculator. This is supported by the fact that many of the ordinary calculators would, by reason of the big numbers, give the answer in a scientific notation for example as 1.009195844 + 14, or 1.009195844E + 14, or 1.009195844 ^{14}; i.e. a number with 15 digits. Comparing this result with the answer given by Class Q reveals that the first 10 digits indicated by the class is correct, and more then that, the answer given by the class contains also the correct number of digits. It seems reasonable to suppose that Class Q did know that the answer had to be a number with 15 digits. Applying a calculator the first 10 of these digits were calculated. The problem was the last five digits. In order to figure out this 15-digit number, Class Q most likely removed the decimal symbol and substituted the five zeros for the digits not shown by the calculator. The calculation of the number of white squares gives further support for the use of calculators, and the adding of zeros. This number was at page 5 calculated to be $100919584400000 - 20091747 = 100919564300000$. It is obvious that this subtraction is incorrect. It looks like Class Q did not reflect on the correctness of these two answers. The relationship between the units digits to the factors in a multiplication and the units digit in the answer is not of the simplest one, it is therefore not surprising that Class Q did not realise the incorrectness of the multiplication answer. It is a bit more surprising that they did not realise the incorrectness of the subtraction answer. However, all of the incorrect numbers are big ones. A question to be posed, but not discussed further in this work is: Does calculation with big numbers make blind?

8.8.3 Summary – Class Q

It is argued that Class Q made use of two methods for determining the number of black, and the number of white squares at the diagonal. The first method relied on mental calculation and was used on relatively small numbers. The second method applied two explicit linear expressions for determining the two numbers in question. This method can be considered to be a formalising of the mental calculation, and the method can therefore be regarded as an extension to the first method. In a mathematical language, the second method used linear functions in one variable.

The total number of squares in a pattern was calculated by a second-degree expression. Mathematically this expression was a composite function of second degree in one variable. Class Q stated also a relationship between the number of squares at the diagonal, and the number of black squares, and the number of white squares in the pattern. This relationship was used to calculate, in an alternative way, the number of white squares in the pattern. Mathematically this is also a composite function of second-degree in one variable.

It is noticed that Class Q, when calculating with big numbers, did not realise the incorrectness of the answers. Why they did not realise this is a question that still is an open question.

Mathematical structures inherent in this solution:

- Figurative numbers (square numbers);
- Explicit defined linear functions, and functions of second degree in one variable;
- Composite function of second degree in one variable.

8.9 Class R

This was the second time Class R participated in a *Tangenten* competition. Their first participation was as Class H, with an answer to the task *A box with ten cubes*, *Tangenten* **1**(2). See section 6.8. This answer differs from the other answers to this task through the way mathematical symbols are used. The stated relationships are, to a large extend, formulated as general algebraic expressions. In these expressions the letter *n* is used as a symbol for the number of squares at the diagonal. Based on these general relationships the number of squares in the 99-figure was calculated. Since Class R in their answer introduced *n* as the variable for the number of squares at the diagonal, this symbol will in this section (section 8.9), be used instead of the symbol *t*, which is used elsewhere in this chapter.

8.9.1 Solution

This solution is introduced by listing the four formulas:

$$n:2+0{,}5=\text{ blacks in the width}$$
$$\text{black}-1=\text{whites in the width}$$
$$\text{blacks}\cdot\text{blacks}=\text{all the blacks}$$
$$\text{whites}\cdot\text{whites}=\text{all the whites}$$

(Class R, p. 2)

How Class R found these formulas is not reported, out over that they wrote they had been discovered. The formulas quoted above, Class R found "a bit bothersome, so we simplified it" (Class P, p. 2). This simplifying resulted in the following expressions:

$$\frac{n+1}{2}=\text{ blacks in the witdh}\qquad \frac{n-1}{2}=\text{ whites in the witdh}$$

In order to find all the black and the white squares we did it like this:

$$\left(\frac{n+1}{2}\right)^2=\text{ all the blacks}\qquad \left(\frac{n-1}{2}\right)^2=\text{ all the whites}$$ (Class P, p. 2)

The number of squares in an arbitrary *n*-figure was then given as the sum of $\left(\frac{n+1}{2}\right)^2$, and $\left(\frac{n-1}{2}\right)^2$. By setting $n=99$ the number of squares in the 99-figure was correctly calculated to be 4901 (Class P, p. 4). It is noticed that this solution is identical with the solution in section 8.3.2, the sum of two squares. Class R did not explicitly state that *n* symbolised the number of squares at the diagonal, but as seen from the quotations there can be no doubt that it is tacitly understood that *n* symbolises this quantity. Nor was the oddity of *n* explicitly stressed. However, it seems very reasonable to suppose that Class R was aware of that fact since they stated that the number of black squares 'in the with' can be calculated with the formulas $n:2+0{,}5$ or $\frac{n+1}{2}$. This is further supported by the fact that all the *n*-values in the table written at page 5 are odd numbers. The formulas stated for determining the number of black and white squares at the diagonal are linear functions. The independent variable is *n*, the number of squares at the diagonal. If the function that determine the number of black squares at the diagonal is called *g* this function can be symbolised by $g(n)=\frac{n+1}{2}$. Correspondingly if *h* symbolises the number of white squares at the diagonal, then $h(n)=\frac{n-1}{2}=g(n)-1$. In order to calculate the total number of squares in the *n*-figure, $A(n)$, Class R squared the outcome of the two linear functions and added the result. If the squaring and adding are called *f*, then the whole process can be symbolised:

$$A(n) = f(g(n)) = (g(n))^2 + (g(n)-1)^2 = \left(\frac{n+1}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2$$

Mathematically this is a composite function of second degree in one variable.

In addition to the formulas Class P has, on page 5, also given a function table for this function. The table range for odd numbers n from $n=1$ to 31. In addition to the values for n and $A(n)$, the table contains columns for the number of black squares at the diagonal, $g(n)$, the number of black squares in a pattern, $(g(n))^2$, the increase of the number of black squares going from one figure to the next, $g(n)-g(n-2)$. Correspondingly columns are presented for the white squares. Class R does not give any direct statement concerning the increase of the black or the white squares.

8.9.2 Relationship

The following relationship was stated by Class R:

> We found out quite accidentally that the difference between the black and white squares, were so many squares it was across where it was widest.
>
> $\left(\frac{99+1}{2}\right)^2 = 2500 \quad \left(\frac{99-1}{2}\right)^2 = 2401$
>
> $2500 - 2401 = 99$
>
> The final formula became then:
>
> $\left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2 = n$
>
> (Class R, p. 2)

As known from chapter 8.3.1 this is a correct relationship. How Class R discovered this relationship was not commented on in their text, out over that it was ‘quite accidentally’. From the quotation it appears that the conjectured hypothesis first was formulated in the Norwegian language. Then it was translated into an algebraic language. The way Class R continued is however very interesting, they carried through an algebraic proof for their conjecture. Their proof was:

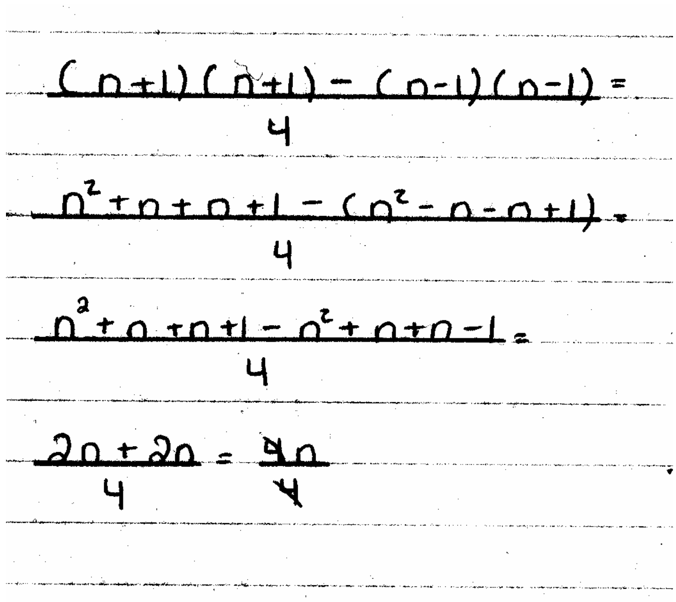


Figure 8.10 The algebraic proof for the conjecture (Class R, p. 3)

It is observed that the proof is correct.

For the total number of squares in a *n*-figure Class Q stated, as commented earlier, that this number was given as the sum $\left(\frac{n+1}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2$, but they did not simplify this sum. They were, however, as emphasised above, able to simplify the difference between the same two second degree expressions. Mathematically the simplifying of these two expressions, the sum and the difference, are very similar, but since the calculation of the difference implies handling a parenthesis with a negative sign, this calculation is more challenging. Taken into consideration that these pupils are 14 - 15 years old the algebraic handling of the difference between the two square numbers demonstrates maturity that is both surprising and promising. An explanation of this maturity is given in a letter from the teacher to the class. Referring to the former competition in which this class participated, he wrote:

> It was very frustrating for me as a teacher, not to be allowed to intervene and direct (the pupils) to the right track, but that was covered up later on (after the pupils answers was mailed) when we examined the problem. We considered the use of n as a "name" of a variable within the natural numbers, how to construct and handling a formula in order to get it as simple as possible.
> This has obviously given results this time. (Class R, p. 1)

What is also found remarkable is that Class R realised the necessity to carry through an algebraic proof for their conjecture. This is stated very clear by the class in the ingress to the algebraic proof, where they emphasises that the calculation is necessary in order to determine if the statement is valid for all numbers. It seems therefore that they have realised that a proof is necessary, it is not enough to 'prove' the validity to a statement for all numbers, by showing the correctness to the statement for some or many number examples.

The sum of two squares given by Class R is commented on by their teacher, in his letter he wrote “It was a pity that they did not manage to simplify their last formula for the sum of all the squares, but ended with two parenthesis expressions in second degree.” (Class R, p.1). Looking at the algebraic handling executed elsewhere in their solution it is very likely that Class R had the necessary capacity to carry through algebraic simplification that could have given the formula (8.6). Despite of the similarity in the calculations of the two expressions, the sum and the difference, the context related to each of them is however different. For the sum Class R had no idea of which algebraic expression the calculation should end up with; they did not know the goal, while for the difference they know or had a very strong feeling of what the result should be; they know the goal.

8.9.3 Summary – Class R

This class introduced *n* as a variable for the length of the diagonal. It is argued that they applied linear functions for determining the number of squares at the diagonal, for each of the two colours, in a *n*-figure. The outcome of these functions, the function-values, was then used as variables for a second degree function. In a mathematical language they used a composite function. The function was given as an explicit algebraic expression.

Class R also discovered and proved a relationship between the total number of black and white squares in a figure, and the number of squares at the diagonal. In a mathematical context this relationship was the well known that the difference between square number *n* and square number $n-1$ equals the nth odd number. This is a relationship between figurative numbers.

Mathematical structures inherent in this solution procedure are:

- Figurative numbers;
- Explicit defined linear and second degree functions in one variable;
- Composite functions of second degree in one variable.

8.10 Summary – Black and white squares

The task invited the pupils to investigate a pattern made up of black and white squares. They were explicit asked to find the total number of squares in a relatively large pattern, the 99-figure, and also to look for alternative procedures to determine that number. Another challenge was to look for other relationships.

One of the answers, Class M, is different from the other five answers this answer was based on systematic explorations of the smallest figures. These explorations produced numbers which were put in a table and then the table was investigated. The outcome of these investigations resulted in several relationships; some of the relationships were logical equivalent. By applying three of these relationships the number of squares in the 99-figure was fixed. Two of the relationships applied for this calculation were first order linear recurrence relations; the third was an explicit expression.

The procedures used by the five remaining classes were based on geometrical decomposition; three of these had in fact solutions which were based on two different geometrical decompositions. The relationships stated were second degree explicit expressions in one variable. One class, Class P, discovered and described also a second relationship for determining the number of squares in a pattern. Most likely this discovery was not found upon geometrical considerations. The described relationship was also a second degree explicit expression in one variable.

One challenge was to look for relationships not only related to determine the number of squares in a figure. One such relationship was the relation between two consecutive square numbers and the odd numbers, or the equivalent relationship between the sum of the first *n* odd numbers and the square numbers. Five of the classes described this relationship, Class M, applied in fact this relationship as one of the three used in their solution procedure. One of the classes, Class R, gave an algebraic prove for this statement.

Class P produced also a spreadsheet, which gave an overview of the number of squares for patterns where the length of the diagonal range from 1 to 113. In the construction of the spreadsheet it is argued that one of the columns is defined by a first order linear recurrence relation, while most likely the remaining cells were defined explicitly.

An overview of the mathematical structures inherent in the solution procedures to the classes is given in the following table:

Class / Mathematical structure	M	N	O	P	Q	R
Recursive defined functions	X			X		
Explicit defined functions	X	X	X	X	X	X
Induction principle	X					
Composition of functions		X	X		X	X
Figurative numbers		X	X	X	X	X

Table 8.2 Overview of the mathematical structures inherent in the solution procedures to the task *Black and white squares*

9. ARITHMOGONS

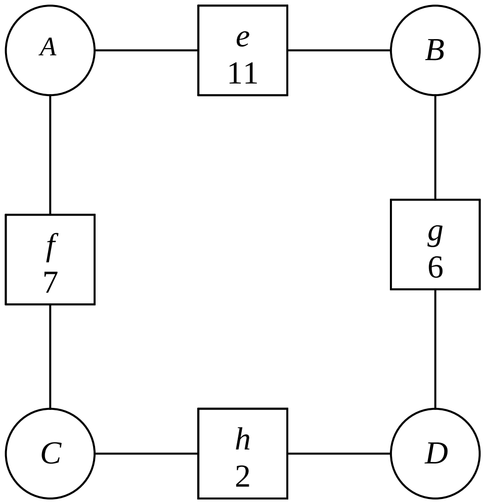
Only one class responded on this task

Class S: A grade 5 class.

9.1 The task

Squares

Complete the 'arithsquare' after the following rule: The number in a square (box) equals the sum of the numbers in the circles on either side of it.



Is there more than one solution?
The numbers in a solution shall be integers greater than or equal to zero.

Put your result in a table.

Below you will find some other numbers, which you shall use in the squares (boxes).
In each case, try to find all the possible solutions.

Choose your own numbers for the squares (boxes), and find all the possible solutions.

Study your solutions and look for a system (relationship). Can you find a rule for the number of solutions in each of the cases?

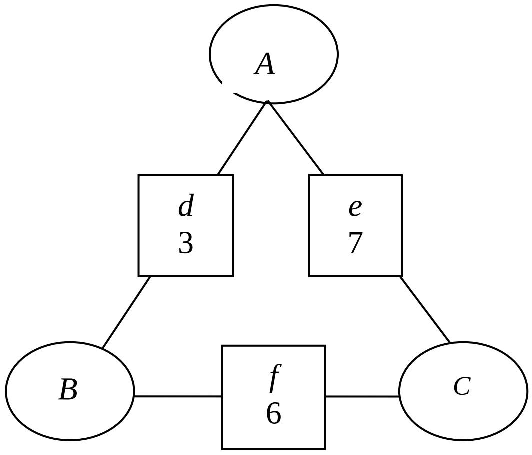
Have you observed or discovered a pattern or a relationship?

Write about what you have done, and your findings.

1. $e = 4$ $f = 3$ $g = 7$ $h = 6$
2. $e = 3$ $f = 8$ $g = 5$ $h = 10$
3. $e = 7$ $f = 3$ $g = 4$ $h = 2$
4. $e = 15$ $f = 9$ $g = 12$ $h = 6$
5. $e = 2$ $f = 5$ $g = 6$ $h = 8$

Triangles

Complete the 'arithtriangle' after the same rule as before: The number in a square (box) equals the sum of the numbers in the circles on either side of it.



Is there more than one solution?
The numbers in a solution shall be integers greater than or equal to zero.

Do the same with the arithtriangle as you did with the arithsquare. Use for example the numbers

1.	$d = 4$	$e = 1$	$f = 3$
2.	$d = 5$	$e = 3$	$f = 4$
3.	$d = 8$	$e = 6$	$f = 4$
4.	$d = 8$	$e = 3$	$f = 3$
5.	$d = 9$	$e = 6$	$f = 8$

Choose also your own numbers.

Compare the results for the arithsquare and the arithtriangle. Write about your findings.

We shall make an extension to this problem. We will also accept fractions in a solution. The numbers in the squares (boxes) are still integers greater than or equal to zero.
What are your findings?

If we also accept negative numbers in a solution, what do you discover?
The numbers in the squares (boxes) are still integers greater than or equal to zero.

Extension of the problem

A natural extension is to examine the arithpentagon, arithhexagon etc., and compare the results for the different arithmogons. Interesting relationships can be discovered.

9.2 Solution

9.2.1 Arithsquares

An arithsquare contains eight numbers, four of them, *e*, *f*, *g*, and *h* have been given, and the remaining four, *A*, *B*, *C*, and *D*, shall be determined according to the given rule, which says that the sum of the numbers in two neighbouring circles equals the number in the square lying between. The overall question for this task is whether it is possible, for all combinations of the four numbers *e*, *f*, *g*, and *h*, to determine the four numbers *A*, *B*, *C*, and *D*; i.e. to always find a solution. The given rule implies a system of four equations with four unknown:

$$A + B = e \text{ , } A + C = f \text{ , } B + D = g \text{ , and } C + D = h \tag{9.1}$$

The two equations $A + B = e$ and $C + D = h$ implies $A + B + C + D = e + h$, while the other two equations $A + C = f$ and $B + D = g$ gives $A + B + C + D = f + g$. A necessary condition for solution is therefore:

$$e + h = f + g \tag{9.2}$$

The next question is then if (9.2) is a sufficient condition for solution? The answer to this question is yes. A solution in non-negative integers can for example be constructed in the following way: It is no restriction to suppose that *e* is the smallest of the numbers *e*, *f*, *g*, and *h*; i.e. $e = \min(e, f, g, h)$. Give this value to one of the neighbouring circles to the *e* box, for example *A*. Then $A = e$, $B = 0$, $C = f - e$, and $D = g$ is a solution in non-negative integers. The above can be formulated in the following proposition:

> An arithsquare has solution if and only if the sums of the numbers in opposite squares are identical.

As a consequence of this proposition the examples 3. and 5., given in the text, have no solution.

The next question is then, if solution, how many solutions in non-negative integers does exist? Suppose that the arithsquare has solution; i.e. $e + h = f + g$. Above is shown how one solution can be constructed. In order to fix the number of solutions in non-negative integers, this solution will be the basis for the construction of all the non-negative solutions. Set $e = \min(e, f, g, h)$, and let *j* be an integer that satisfies $0 \leq j \leq e$. Then let $A = e - j$, $B = j$, $C = f - e + j$, and $D = g - j$. Each value of *j* will give a solution, and consequently there will be at least $e + 1$ different solutions. It remains to argue that these $e + 1$ solutions are the only solutions in non-negative numbers. From above it is observed that for these solutions the values of *A* range from 0 to *e* inclusive, and that these values are the only possible non-negative values for *A*. Suppose that there are more than $e + 1$ solutions. This means that at least two different solutions have identical *A* value, and that they must differ on at least one of the three remaining values *B*, *C*, or *D*. From the equations (9.1) it follows that this is impossible. This result can be summarised as:

> If an arithsquare has non-negative solution(s), the number of solutions is given by $\min(e, f, g, h) + 1$.

9.2.2 Arithtriangles

Quite analogue to the previous figure, the arithsquare, the rule gives in this case a set of three equations with three unknowns. Using the notation from the figure these equations are:

$$A + B = d \text{ , } A + C = e \text{ , and } B + C = f \quad (9.3)$$

This system of equations has a unique solution given by:

$$A = \frac{d + e - f}{2} \text{ , } B = \frac{d - e + f}{2} \text{ , and } C = \frac{-d + e + f}{2} \quad (9.4)$$

If the values *A*, *B*, and *C* shall be non-negative integers it is observed that the biggest of the three numbers *d*, *e*, and *f* must be equal or less than the sum of the two other numbers, and additionally either all three numbers must be even numbers or exact on of them must be an even number. If it is accepted that the numbers *A*, *B*, and *C* can be fractions, (9.4) gives that there always will be a unique solution, and thus the system of equations is determined.

Another relationship that directly follows from (9.4) is:

$$A + f = B + e = C + d = \frac{d + e + f}{2} \quad (9.5)$$

Using words this relationship tells that the sum of a number in a circle, one unknown, and the number given in the box located on the opposite side to the circle, always equals half of the sum to the three given numbers.

9.2.3 Generally

There is a marked distinction between the results obtained for these two figures, the arithsquare and the arithtriangles. This distinction becomes more visible if the numbers used in the figures are not restricted only to non-negative integers, but it also acceptable to use rational numbers.

In this case the arithsquare has either no solutions or infinitely many, which corresponds that the system of equations, (9.1), is either self-contradicting or undetermined. For the case of the arithtriangle there is one unique solution. The system of equations, (9.4), is determined. This can be further extended to arithmogons with *n*-sides, and the result is that if the number of sides, *n*, in a figure is odd then there is a unique solution, while if it is even there is either no solution or infinitely many solutions (McIntosh & Quadling, 1975).

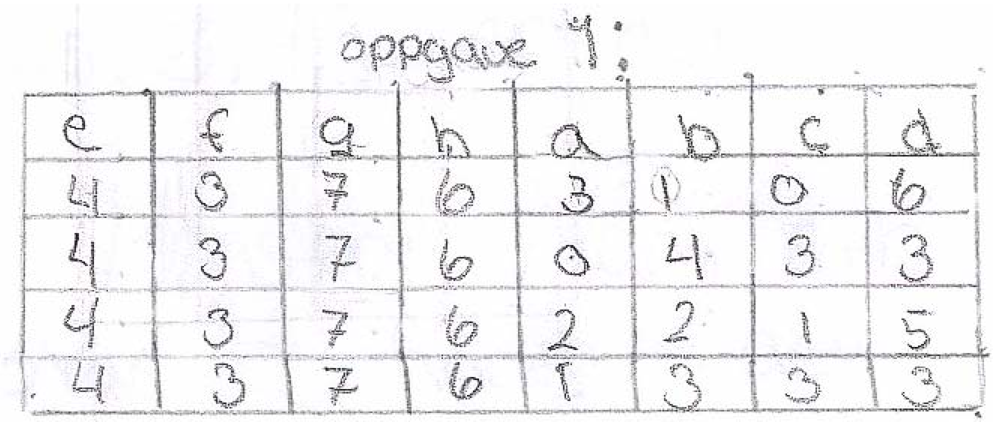
Wittmann (1995) mention arithmogons as an example of what he called ‘a substantial teaching unit’, and he maintains “The mathematics behind arithmogons is quite advanced” (p. 367). He exemplifies this advanced mathematics with arithtriangeles. The number in the squares (boxes) *d*, *e*, and *f*, form a vector (d, e, f), the same is the case for the numbers in the circles *A*, *B*, and *C*, (A, B, C). The given rule, adding the numbers in the circles, defines the mapping $(A, B, C) \to (d, e, f)$, which is a linear mapping from a three-dimensional vectorspace over the rational numbers (real numbers), into itself. Since the corresponding matrix, which in this case is $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is non-singular the vector (A, B, C) is fixed. This is generalised by McIntosh & Quadling (1975) to *n*-gons, arithgons with *n* sides. For an *n*-gon the rule defines a mapping from an *n*-dimensional vectorspace over the rational- or real-numbers into itself. If *n* is odd the corresponding matrix is non-singular and there is a unique solution, if *n* is even the matrix is not non-singular which means that there will be either no or an unlimited number of solutions. In the *n* even case the given numbers have to satisfy a condition in order to have solutions. For $n = 4$ the condition is given by (9.2).

9.3 Class S

For the remaining part of this chapter, if nothing else is stated, a number means a non-negative integer, and a solution of/to an arithmogon consists of such numbers. For both the arithsquare and the arithtriangle the text gave one number example for each of the two figures, and in addition, for each figure, it was also suggested five different set of numbers the classes could choose in their explorations. These sets Class S has, for each of the two figures, numbered: problem 0 (the numbers given in a figure), problem 1, problem 2, …, problem 5 (Class S, p. 1).

9.3.1 Arithsquares

Class S in their answer used the numbers given in the text. For those arithsquares that did not have a solution, problem 3 and 5, they shortly stated impossible. For those who have solution, their results were, as suggested in the text, presented in tables. For example the table for problem 1:


oppgave 1:

e	f	g	h	a	b	c	d
4	3	7	6	3	1	0	6
4	3	7	6	0	4	3	3
4	3	7	6	2	2	1	5
4	3	7	6	1	3	3	3

Figure 9.1 Table constructed by Class S (p. 1)

It is observed that two of the lines in the tables contain errors. Problem 1, above, the last line the table above states that $a = 1$, $b = 3$, $c = 3$, and $d = 3$ is a solution, and for problem 4 they claim that $a = 6$, $b = 9$, $c = 6$, and $d = 3$ is a solution. These errors are probably ‘miss-printings’, the first case for the solution $a = 1$, $b = 3$, $c = 2$, and $d = 4$, and the second one for $a = 6$, $b = 9$, $c = 3$, and $d = 3$. Except from these errors all the solutions given by the class in these tables are correct. Related to the focus of this work these errors are of no significance.

One of the problems of the task was to find all the (possible) solutions for each of the arithsquares. Looking at the given tables it is observed that except for problem 4, the number of possible solutions in each of the cases is correct. In the fourth case Class S has ‘lost’ two solutions.

In connection with the given tables Class S did not explicitly state or assert that the solutions listed in the tables were all the possible solutions in each of the cases. However, from what they wrote at page 2 it is very likely to interpret that it is tacitly understood that each table contains the complete solution set. The main reason for this assumption is that Class S has explicitly stated the number of solutions in each of the cases. Except for problem 2, these numbers corresponds with the number of solutions given in the tables. For problem 2 the number of solutions in the table, four, is correct, while in their statement at page 2 they operates with five solutions. This could be a ‘miss-counting’ or a ‘miss-printing’. The number of lines in the table is five. However, related to the focus of this study, this discrepancy is of no significance. If the number of solutions is greater than zero, Class S did not formulate any ‘general’ statement about the number of solutions. On account of the incorrectness in their overview of the number of solutions, it would, most probably, be difficult to formulate a credible or an acceptable rule.

Linked to the given overview over the number of solutions, they added the sentence or statement “When it is two digits one after another (ex. 3,4,), it is no solution.” (Class S, p.2). The interpretation of this statement is, as far as can be seen, not straightforward. There are two factors that needs to be discussed, firstly the use of the word ‘digits’, and secondly what is found in the parenthesis, ‘ex. 3,4,’.

In the statement Class S used the corresponding Norwegian word for ‘digits’. The context “it is two digits one after another”, indicates strongly that the word digits here is used in the meaning numbers. There are at least two arguments for this interpretation. Firstly, as is well known that a natural number less than ten in our number system is written as a ‘one sign object’, a one-digit symbol, and it is therefore not surprisingly that pupils at grade 5 mix the concepts digit and number. A one-digit symbol can represent a number, and a number can be represented by a one-digit symbol. Secondly, is the fact that for every integer greater than nine, the digits are “one after another”. It is not likely that Class S meant that if an arithsquare should have solution, none of the given numbers could be greater than nine; i.e. none of the given numbers could have two digits or more, even though this was the case for problem 3 and problem 5, the two problems that did not have solution.

Then there is the challenge with what has been written in the parenthesis, ex. 3, 4, what do these two numbers refer to? The location of this sentence indicates strongly that the intention is to explain why the two given examples, problem 3 and problem 5, did not have solution. This intention is supported by the fact that Class S later on in their answer explicitly stated a general rule for the no solution case. The two numbers 3 and 4 are given in problem 3 so one possibility is that these numbers refers to this particular arithsquare. This interpretation is supported by the fact that Class S has stressed that this is an example; i.e. tacitly understood an example supporting their statement. Another possible interpretation is that these two numbers, 3 and 4, refers to the numbering of the problems; i.e. are the co-ordinates to the none-solution problems. This interpretation is found to be less probable than the first one, mainly because problem 4 has solution and none of the given numbers *e*, *f*, *g*, and *h* in that example are 'one after another'. Irrespective of which of the two interpretations are correct, the statement to Class S can most likely be formulated as:

> If two of the numbers *e*, *f*, *g*, and *h* are one after another then an arithsquare has no solution.

It is found a bit surprisingly that Class S has formulated this in-correct statement, this mainly because that both problem 0, and problem 1 have solution and two of the numbers are one after another.

At the end of the section arithsquare Class S formulated two relationships. For the solution case they stated "Pattern: When $e+h$ and $f+g$ gives the same answer there exists a solution." (Class S, p. 2). For the case of no solutions, Class S stated one main rule "If $f+g$ not gives the same sum as $e+h$ there exist no solution." (Class S, p. 2). As seen from paragraph 9.2.1 both of these two relationships are correct.

The statement related to the no-solution case is followed by an explanation for why the two problems 3 and 5 do not have solutions. Their argument was "no. 3 is too small and no 5 is the 2 number too small" (Class S, p. 2). The meaning to this sentence is not obvious. Since they here refer to two problems, problem 3 and problem 5, it is found naturally to separate this sentence into two parts, one for each problem. The first part is then "no. 3 is too small". Isolated from the context this sentence or phrase is without meaning, no. 3 is a problem, and it is hardly to interpret that a problem could be 'to small'. What is then 'too small'? In this context it is reasonable to suppose that the intention to Class S was to state that some, one or more of the given numbers are too small; i.e. at least one of the four numbers $e=7$, $f=3$, $g=4$, or $h=2$ are to small. Taken into consideration that the sentence is an explanation to the relationship "If $f+g$ not gives the same sum as $e+h$ there exists no solution", where $f+g$ is written before $e+h$, and in addition $f+g=7$ and $e+h=9$, one possible interpretation is that at least one of *f* or *g* is too small. This is also supported by the fact that in the second part of the explanation, which relates to problem 5, they explicitly state that it is a particular number that is too small. The second part of the sentence stated that for the problem 5 it was the number 2, the *e*-value, which was too small. In this case $f+g=11$ and $e+h=10$, therefore the statement is so far correct, but it would be equally correct to state that the *h* value was too small. Why they explained the impossibility by arguing that some numbers were to small, and not using that others were too big, which would have been equally correct, is an open question.

In the two rules Class S used the corresponding Norwegian word for 'gives'. The context is that something, in this case the result of comparing two sums, either 'gives' or not 'gives' a result. Each of those two sums was constructed by adding two numbers. The starting point is then four independent numbers, variables. From these variables two sums are constructed, then these sums are compared, the result of this comparison results in one of two values, *solution* or *no-solution*. This can be illustrated by:

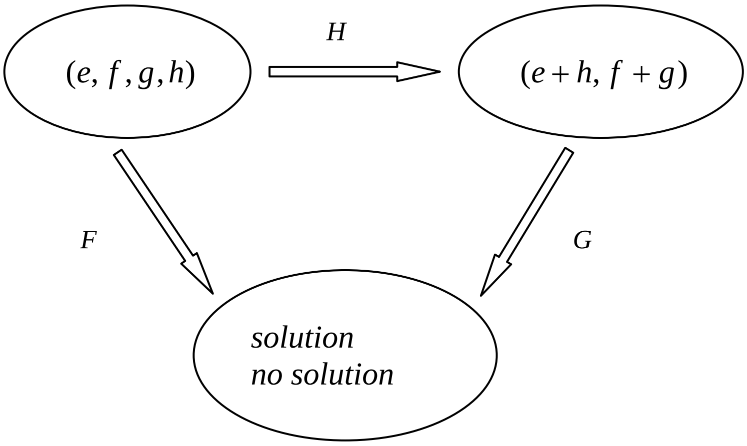


Figure 9.2 Illustration of the relationship stated by Class S

With mathematical symbols this illustration can be described by:

$$H(e,f,g,h)=(e+h,f+g)\ ,\ G(x,y)=\begin{cases} solution & \text{if } x=y \\ no\ \ solution & \text{if } x\neq y \end{cases}$$

and

$$F(e,f,g,h)=G(H(e,f,g,h))=G(e+h,f+g)=\begin{cases} solution & \text{if } e+h=f+g \\ no\ \ solution & \text{if } e+h\neq f+g \end{cases}$$

From the above it is seen that H is a function from $\mathbb{N}_0^4$ to $\mathbb{N}_0^2$, G a function from $\mathbb{N}_0^2$ to $L=\{solution, no\ \ solution\}$, and F a function from $\mathbb{N}_0^4$ to L, where $F=H\circ G$.

9.3.2 Arithtriangles

Analogue to the case of arithsquares, an arithtriangle solution is a non-negative triple of integers, $A, B,$ and C, satisfying the given rules. As in the arithsquare case Class S numbered the given problems from 0 to 5, see paragraph 9.3.1.

Class S presented a correct list over which of the given problems that have solution. The list shows also that the number of solutions to each of the problems is either one or zero. It was not explicitly stated that this could be or would be a general rule, that despite that one of the questions given in the task was if there would be more than one solution for each arithtriangles. However, they stated that the number of solutions in the arithsquare case was bigger than in this case, but if they intended to maintain that this was or could be the case for each separate problem, is found difficult to ascertain. Their formulation was:

> It is more solutions to the square than it is to the triangle.
> It is two numbers on the square and the triangle.
> Number 5 at both the square and the triangle does not work out.
> It is bigger numbers on the square than on the triangle. (Class S, p. 3)

One possible interpretation is; if an arithsquare has solution(s) the number of solutions would be bigger than for an arithtriangle. However, it is also possible that they intended to say that the number of problems in each of the two cases, arithsquare and arithtriangle, is identical, and for both, two of the problems were no-solution problems. For each of the four remaining problems the number of solutions in the arithsquare case is bigger than in the triangular case, and hence the total number of solutions in the former case is bigger than in the last one.

For the two no-solution problems Class S explained or argued, as they did in the arithsquare case, for why this occurred. This is in contrast to the problems that had solution and where they not explicitly stated the solutions. However, that does not necessarily mean that Class S did not calculate these solutions; i.e. the values of *A*, *B*, and *C*. It is found most likely that these solutions were calculated, in one way or another. There are several arguments that support this interpretation. Firstly, Class S has given the correct number of solutions in each of the cases. Based on the result from the arithsquare problems there was no reason to suppose that the arithtriangles should ‘behave’ different; i.e. also in this case some of the problems could have more than one solution. Secondly, the task did not explicitly ask for the solutions, only for the number of solutions. Their arguments for no-solution “It is no solution to nos 4 because the number 8 is to big. Nos 5 the numbers are too much in succession.” (Class S, p. 3).

The argument for no solution to problem 4, “the number 8 is to big”, may look quite parallel to the argument Class S used in the arithsquare case, see page 183. Exploring the arithtriangle with the numbers $d = 8$, $e = 3$, and $f = 3$, (Problem 4) see figure 9.3, and remembering the rules, reveals relatively rapidly that using the biggest possible values for *A* and *B*, which are 3 for each, and the sum of those two numbers could therefore not be 8. So the argument to Class S is quite natural. As known from paragraph 9.2.2 there is a solution to this arithtriangles, in this case the *C* value of the solution is negative, -1.

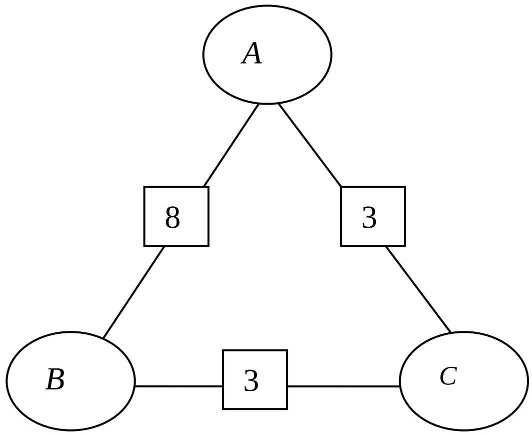


Figure 9.3 Arithtriangle number four

The statement concerning problem 5, “Nos 5 the numbers are too much in succession” is difficult to grasp. As known from paragraph 9.2.2, the reason for no solution is the parity to the three given numbers. Since just one of them is odd there is no integer solution. There is no indication in the answer that Class S has realised this fact.

The task invited also to compare the results for the arithsquare and for the arithtriangle. In the square case Class S stated a relationship between the four given numbers that had to be satisfied if there should be solution, and also a relationship between the same four numbers for the no-solution case (see page 183). Establishing of these two relationships has then possibly ‘inspired’ the class to look for two relationships also in the arithtriangle case, one for the solution case and one for the no-solution. They stated:

> b) If (ex. no 1. $a + f$ is the same sum as $b + e$ then $d + c$ is the same sum.
> c) If $a + f$ not gives the same sum as $b + e$ and $d + c$ it is no solution. (Class S, p. 3).

As some of the former relationships formulated by this class, the language used opens for several interpretations, and in retrospect it can be difficult or impossible to figure out what they intended to tell. It could be remarked that if a grade 5 class had used a written language, in their statements, that had been mathematical correct, one would be suspicious, and question who really had formulated the statements. What is found most likely is that the two stated relationships for the arithsquares, the solution case and the no-solution case, has, as mentioned above, inspired the class to formulate two rules also in this case. On that basis it is reasonable to assert that the first of the two relationships, b), quoted above, is a statement referring to the solution case, and the second, c), to the no-solution case. The first statement does not explicitly refer back to the solution case, as the second statement refers to the no-solution, but the reference to problem 1 in the first statement, a problem that has solution, supports this interpretation. It is therefore found highly probable that the interpretation of the first statement is:

If *A*, *B* and *C* is a solution of the arithtriangle the three sums $A+f$, $B+e$, and $C+d$ are identical.

This is a consequence of relationship (9.5).

The second statement concerns the no-solution aspect. The ending phrase of the statement "it is no solution" carries an ambiguity. What does the 'no solution' term refer to? Is it:

i. the arithtriangle that not has a solution?

or

ii. that *A*, *B*, and *C* is not a solution?

If it is the first one then the intention to Class S could be to state:

If it is impossible to find numbers *A*, *B* and *C* such that $A+f=B+e=C+d$ then it is a no-solution arithtriangle.

The second one:

If not $A+f=B+e=C+d$ then *A*, *B*, and *C* is not a solution. Logically this is equivalent with the solution statement.

This last one does not imply that the given arithtriangle is a no-solution triangle. As known from paragraph 9.2.2 both of these statements are correct.

On page 185 it is argued that Class S on the basis of the given number triples, *d*, *e*, and *f*, in one way or another, has calculated the solutions *A*, *B*, and *C*. Mathematically this means that they have used a function. In this case a function from $\mathbb{N}_0^3$ to $L=\{solution, no\ \ solution\}$.

As could be expected, this is quite parallel to the arithsquare case. However, comparing these two cases reveal possibly a difference. In the first case, the square, it is argued that the calculation of the *L*-values is based on comparing two intermediate calculated values. For the arithtriangle it seems that the *L*-values have been worked out directly from the given values.

9.4 Summary – Arithmogons

Only one class responded to this task. It is argued that they in the solving procedure applied explicit defined functions with respectively four and three independent variables and composition of functions.

10. IN THE CITY

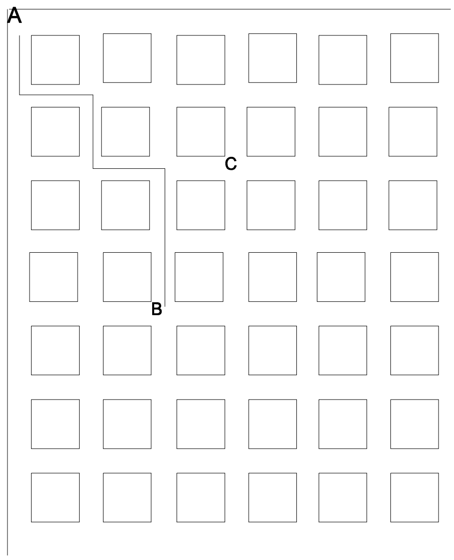
Four classes participated in the competition:

Class T: A grade 3 class;
Class U: A grade 4 class;
Class V: A grade 8 class;
Class W: A grade 9 class.

The classes T, U, and V participated all for the second time.

10.1 The task

Anne is living in a city where the streets make a squared network, and where all of the houses are of equal size. Look at the figure (map).



Anne lives in the house marked with A, she lives in the house at the corner. Berit her friend, lives at B. Each time Anne is going to visit Berit she follows the same way. This way is marked on the map. One day while she is on her way to Berit along the usual way, she decided that next time she will walk a different way.

Examine how many different ways she can choose from. She shall always follow a way, which is as short as possible. (This way shall therefore always have the same length as the usual way between A and B.)

How many different and shortest ways are there from A to C?

Examine how many shortest ways there are from A to all of the street-crossings in a moderate distance from A. Arrange these numbers in a manner, which is easy to grasp.

Inspect these numbers and look for relationships. Write down the relationships you discover.

NB! Anne keeps always the traffic regulations she will always cross the streets straight ahead, never diagonally.

This problem can be extended towards the probability concept.

We can imagine that Anne chooses her way by chance; every time she comes to a street-crossing she tosses a coin. (She will only have two possibilities because she shall not move backwards.) What is the probability that she arrives at Berit, if she starts at A?

Another possibility for an extension is random choices.

Try to write down one sentence, not too long, which gives a complete description of a shortest way from A to B. Your description should make it possible for a (random) person to choose by himself, at random a (shortest) way.
Try also to put this in relation to the original problem.

10.2 Solution

This task or variants of it is found in both mathematical textbooks (Biddle, Savage, Smith & Vowles, 1988; Grimaldi, 1994; Stowasser & Mohry, 1978) and mathematical education books (Freudenthal, 1978; Polya, 1981).

In order to refer to the different road crossings in a city it is practical to either place the city map in a co-ordinate system or a co-ordinate system in the city map. For this particular city it is found expedient to locate the origin in the upper left corner; i.e. where *A* is located. An ordered couple, (h,v), where *h* is the number of blocks to the right of *A*, and *v* is the number of blocks below *A*, can then identify every crossroads in the city. The co-ordinate to the three given crossroads is then $A=(0,0)$, $B=(2,4)$, and $C=(3,2)$. The different crossroads can also be identified as lattice points. Further it turns out to be expedient to have separate names for the crossroads in the inner part of the city, and those at the upper and the left side. The crossroads at the inner part will be called ×-crosses, the other T-crosses. Mathematically a ×-cross has co-ordinate (h,v) where $h \cdot v \neq 0$, and a T-cross has co-ordinate (h,v) where $h \cdot v = 0$. For the rest of this task, a way means a shortest way.

10.2.1 The number of ways

Let $W(h,v)$ be the total number of ways between $A, (0,0)$, and an arbitrary cross, (h,v). The number of ways in the city can then be organised in a table like:

A	$W(1,0)$	$W(2,0)$	$W(3,0)$	$W(4,0)$	$W(5,0)$	$W(6,0)$	$\dots$
$W(0,1)$	$W(1,1)$	$W(2,1)$	$W(3,1)$	$W(4,1)$	$W(5,1)$	$\vdots\ \dots$	
$W(0,2)$	$W(1,2)$	$W(2,2)$	$W(3,2)$	$W(4,2)$	$\vdots\dots$		
$W(0,3)$	$W(1,3)$	$W(2,3)$	$W(3,3)$	$\vdots\dots$			
$W(0,4)$	$W(1,4)$	$W(2,4)$	$\vdots\dots$				
$W(0,5)$	$W(1,5)$	$\vdots\dots$					
$W(0,6)$	$\vdots\dots$						
$\vdots$							

Table 10.1 Number of ways

If table 10.1 are completed with numbers some more or less familiar number patterns appears:

- all the numbers in the first line and the first column are identical, i.e. they are all 1;
- the second line and the second column are the natural or the counting numbers;
- the third line and the third column consists of the triangular numbers;
- a number not in the first line or the first column is the sum of the nearest number to the left and the nearest number above.

For the number of ways from *A* to *B* it is not difficult to figure out that $W(2,4)=15$ and $W(3,2)=10$; i.e. there is 15 shortest ways from *A* to *B*, and 10 from *A* to *C*.

From above the number of ways to a T-cross is one. For an ×-cross, *P*, the total number of ways is the total number of ways to the nearest cross to the left of *P* added to the number of total ways to the nearest cross above *P*.

In this context the number of ways from *A* to *A*, that could be symbolised as $W(0,0)$, is in a sense, not an interesting question, however it turns out to be convenient to define or associate a number to this quantity. It appears that defining $W(0,0)=1$ is reasonable. The above relationship can then be symbolised:

$$\begin{aligned} &W(h,0)=W(0,v)=1 \text{ where } h\geq 0 \text{ and } v\geq 0; \\ &W(h,v)=W(h-1,v)+W(h,v-1) \text{ if } h>0 \text{ and } v>0 \end{aligned} \qquad (10.1)$$

This is a recursive relationship. It is also well known that the relationship (10.1) and hence the number of ways, table 10.1, turns out to be the numbers known as *Pascal's* triangle.

10.2.2 Pascal's Triangle

It is a well known fact that this number pattern contains many relationships, and this makes Pascal's triangle a very rich source for investigations. A frequently used way to present Pascal's triangle is:

$$\begin{array}{ccccccccccccc} &&&&&&1&&&&&& \\ &&&&&1&&1&&&&& \\ &&&&1&&2&&1&&&& \\ &&&1&&3&&3&&1&&& \\ &&1&&4&&6&&4&&1&& \\ &1&&5&&10&&10&&5&&1& \\ 1&&6&&15&&20&&15&&6&&1 \\ \dots&&&&&&&&&&&&\dots \end{array}$$

Figure 10.1 Pascal's Triangle

The horizontal rows in the Pascal triangle are usually called lines, and the oblique lines for diagonals. A diagonal that has direction from the left side of this triangle down to the right side will be called a right-diagonal; correspondingly a diagonal that crosses a right-diagonal is a left-diagonal.

Among the relationships inherent in Pascal's triangle are:

> Pascal's triangle is symmetrical with respect to the vertical line through the top of the triangle, the top number 1. (10.2)

> Starting at the top of a diagonal, at the number 1, and add up all the numbers on this diagonal to an arbitrary line, then the answer is a number in the next line. The co-ordinate to this number depends on what type of diagonal. If the added numbers were on a left-diagonal, the answer is on the next left-diagonal, if they were on a right-diagonal it is on the previous right-diagonal. (10.3)

> Adding up all the numbers in a line is always a power of two, or to be more precisely the power of two raised to number of the line. (10.4)

For further relationships in this number pattern see for example Enzensberger (1997), Kirfel (1992) or Polya (1981). Using the symbol for the number of ways these three relationships can be written:

$$W(h,v) = W(v,h) \qquad (10.2)$$

$$W(h,0) + W(h,1) + W(h,2) + \ldots + W(h,n) = W(h+1,n)$$

or (10.3)

$$W(0,v) + W(1,v) + W(2,v) + \ldots + W(n,v) = W(n,v+1)$$

$$W(0,n) + W(1,n-1) + W(2,n-2) + \ldots + W(n,0) = 2^n \qquad (10.4)$$

The symmetry relationship, (10.2), follows directly from the fact that the number of ways if one 'walks' *h* blocks up the street (horizontally to the right), and *v* blocks down the street (vertically downwards) is identical with the number of ways if one 'walks' *v* blocks up the street, and *h* blocks down the street.

The second relationship, (10.3), is also called *Christmas Stocking Theorem* (Butterworth, 1978). The reason for this peculiar name is reflected or found in the figures below:

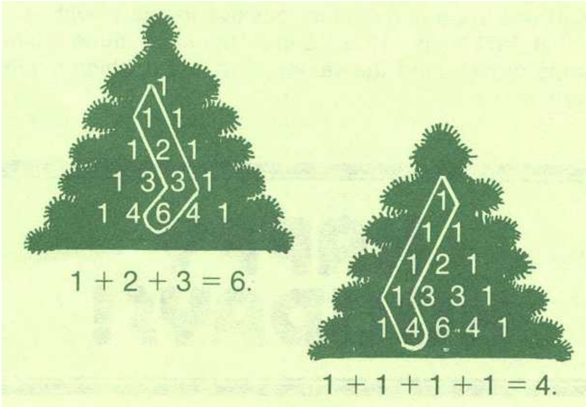


Figure 10.2. Christmas Stocking Theorem. From Butterworth (1978)

A proof for this relationship follows directly from the construction of the triangle.

The last relationship, (10.4), follows also directly from the construction of the triangle. The sum of all the numbers in one line is the double of the sum of all the numbers in the previous line, and since the starting sum, the sum of the number(s) in line 0 is $1 = 2^0$, the relationship is correct.

It turns also out that the numbers in Pascal's triangle are identical with the *binomial coefficients*. If the lines in Pascal's triangle are symbolised by the letter *n* and the right-diagonals by *r*, then the number at line number *n* and diagonal number *r* is symbolised as $\binom{n}{r}$.

Using this symbol the Pascal triangle can be written:

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

…………………………………………………………………………….

Figure 10.3 Pascal's Triangle as binomial coefficients

The relationship between the number of ways and the binomial coefficients is given by:

$$W(h,v) = \binom{h+v}{h} = \binom{h+v}{v} \qquad (10.5)$$

Using binomial coefficients the relationship (10.1) can be written:

$$\begin{aligned} &\binom{n}{0} = \binom{n}{n} = 1 \quad \text{where} \quad n \geq 0 \\ &\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad \text{if} \quad n > r > 0 \end{aligned} \tag{10.6}$$

The relationships (10.2) - (10.4) can be symbolised:

$$\binom{n}{r} = \binom{n}{n-r} \tag{10.2}$$

$$\begin{aligned} &\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \ldots + \binom{n+r}{r} = \binom{n+r+1}{r+1} \\ &\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \ldots + \binom{n+r}{n} = \binom{n+r+1}{n+1} \end{aligned} \tag{10.3}$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n \tag{10.4}$$

It is well known that an explicit expression for $\binom{n}{r}$ is given by:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \tag{10.7}$$

To discover the explicit relationship (10.7) directly from the numbers in Pascal's triangle, figure 10.1 is an arduous task. However, for some of the diagonals it is possible to make explicit descriptions, particular for diagonal nos. one and two. At page 190 two such relationships have been noticed, the counting numbers in diagonal one and the triangular numbers in diagonal two.
The numbers in the two diagonals numbered one can then be expressed by:

$$W(1,v) = \binom{1+v}{1} = v+1 \quad \text{or} \quad W(h,1) = \binom{h+1}{h} = h+1 \tag{10.8}$$

The numbers in a diagonal numbered two is half of the product of two consecutive numbers in diagonal number one; i.e. the numbers in the second diagonals are the triangular numbers. For the right-diagonal this can be expressed:

$$W(2,v) = \frac{W(1,v) \cdot W(1,v+1)}{2} = \frac{(v+1) \cdot (v+2)}{2} = T_{v+1} \tag{10.9}$$

for a left-diagonal the expression is:

$$W(h,2) = \frac{W(h,1) \cdot W(h+1,1)}{2} = \frac{(h+1) \cdot (h+2)}{2} = T_{h+1} \tag{10.10}$$

Another well-known relationship is the link or connection between the numbers in Pascal's triangle and combinatorial analysis. An unordered sample of *r* items drawn from a population of *n* items is identical with $\binom{n}{r}$, these numbers are also known as the *number of combinations* of *n* items taken *r* at a time.

10.2.3 Probability

It was mentioned in the text of the task that one possible extension was towards the probability concept. The suggested problem was: What is the chance or the probability that Anne will arrive at B if she tosses a coin each time she had to choose either to go up the street or to go down the street?
Let $p(P)$ be the probability to arrive at $P=(h,v)$ if the 'tossing a coin' procedure is practised. In this context, and supposing a uniform probability model, the calculation can be done in the following way:

In order to arrive at $B=(2,4)$ it is necessary to toss a coin six times. This activity has $2^6 = 64$ different outcomes. Since there are $W(2,4)=\binom{6}{2}=\binom{6}{4}=15$ shortest ways from *A* to *B* the probability to arrive at *B*, $P(B)$, is given by $\frac{15}{64} \approx 0,24$.

Generally, the probability to arrive at an arbitrary cross $P=(h,v)$ by tossing a coin at each cross is given by:

$$p(P) = p(h,v) = \frac{W(h,v)}{2^{h+v}} = \frac{\binom{h+v}{h}}{2^{h+v}} = \frac{\binom{h+v}{v}}{2^{h+v}} \tag{10.11}$$

Formula (10.11) is an explicit relationship for calculating $p(h,v)$. It is also possible to set up a recursive relationship. Arguments for such a relationship can be:

In order to arrive at $P=(h,v)$, where $h>0$ and $v>0$, one has to come either from $(h-1,v)$ or $(h,v-1)$. The probability to arrive at those to positions are respective $p(h-1,v)$ and $p(h,v-1)$. Tossing a coin has two outcomes, which implies that the probability to arrive at *P* can be calculated as:

$$\begin{aligned} p(h,v) &= \frac{p(h-1,v)}{2} + \frac{p(h,v-1)}{2} \\ &= \frac{1}{2}\left(\frac{W(h-1,v)}{2^{h-1+v}} + \frac{W(h,v-1)}{2^{h+v-1}}\right) \\ &= \frac{W(h,v)}{2^{h+v}} \end{aligned} \tag{10.12}$$

10.3 Remark

The subject for this task was the number of different shortest ways from a corner in a grid system, and the probability related to arriving at a fixed position tossing a coin; i.e. Pascal's triangle. It is however also possible to focus on the length of the different ways in a grid system. This is the basis or starting point for *taxicab geometry*, a subject belonging to *non-Euclidean* geometry; see for example Gardiner (1997), Krause (1986).

10.4 Class T

This text was worked out over a period of four weeks (Class T, p. 1), and the result is a very comprehensive document. It is found appropriate to separate the answer in three main parts that may be called:

1. The number of shortest ways;
2. The probability to arrive at *B*;
3. Drawings.

In addition to the pupils' answer their teacher had attached a detailed written description, organised in ten points, of the working process to the class. This description shows that the teacher organised the activities and also structured the task more detailed than in the original text (Class T, p.1).

10.4.1 The number of ways

Firstly, and in line with one of Polya's (1957) *heuristic strategies*, the pupils studied a similar but simpler problem. They jointly investigated, by drawing on the blackboard, how many shortest ways it was to some of the nearest crosses to *A*; i.e. (1,1) and (2,2). In order to get a more comprehensive data basis each pupil then got a map of the city, and on those maps they draw the ways to some of the other nearest crosses to *A*, among those both *B* and *C*. The results of this activity, the number of ways to these crosses, were then written in a table constructed by the teacher.

It is observed that in this table the values for the shortest ways to the T-crosses have been omitted. Why this has been done is found to be of little interest for the focus of this research. As known from section 10.2 the number of ways to all the T-crosses is 1, which is easily discovered and as such of minor interest for the investigational activity. The drawings that followed this answer showed that the pupils were aware of the fact that the number of shortest ways to the T-crosses is one.

In point 5 (in the teachers description) (Class T, p. 1) the teacher has described how the class continued their activity. The pupils investigated the table and looked for 'mystical things'. They found various relationships, which they dictated to the teacher, and he wrote it down. This methodology was the same as he used in the teaching of their mother tongue. Later on the pupils were organised in groups of two and two. One pupil dictated and the other wrote. One of these dictations is:

> Above stands 2 – 3 – 4. Maybe will it be standing 5 – 6 – 7. Where it stands 2 houses up the street and two houses down the street is it 6 shortest ways. 2 houses up the street and

> 1 house down the street stands it 3. Diagonally down the street where it stands 1 house up the street and 2 houses down the street, stands it 3. 3 plus 3 is 6 and that stands on 2 houses up the street and two houses down the street.
> Above 6 stands it 3. As far as 6 to the left stands 3, 3 and 3 becomes 6. (Class T, p. 3)

In this quotation is found two different relationships, the first is expressed in the first two sentences. Using the symbol from section 10.2 the first sentence can be written:

$$W(1,1)=2 \;,\; W(2,1)=3 \;,\; W(3,1)=3$$

The next sentence "Maybe will it be standing 5 – 6 – 7", indicates that the pattern described by this group is a 'counting on'. The next numbers in this row is then given as:

$$W(4,1)=W(3,1)+1,\; W(5,1)=W(4,1)+1, \text{ etc.}$$

This is further supported from what another group wrote: "On the first row stands it 2 – 3 – 4 – 5 and so on. We count by 1 at a time." (Class T, p. 4). Then the numbers in the first row in their table is given by:

$$W(h+1,1)=W(h,1)+1 \tag{10.13}$$

This is a recursive defined function in one variable, and since a starting value is given, $W(1,1)=1$, the description of the numbers in row one is complete.

The second relationship expressed in the above quotation can be symbolised by:

$$W(2,1)+W(1,2)=W(2,2) \tag{10.14}$$

This indicates that Class T discovered that the numbers inside the table, numbers not in the first row or first column, is given as the sum of the number to the left of the number and the number above. The relationship (10.14) can then be written:

$$W(h-1,v)+W(h,v-1)=W(h,v) \tag{10.15}$$

That Class T has realised that relationship (10.15) is valid is heavily supported from what the class has written on the pages 4-6 and 10. For example:

> On the second row stands 3 – 6 – 10 – 15 and so on. 3 and 6 it becomes 3 between. 6 and 10 it becomes 4 between. 10 and 15 it becomes 5 between. 15 and 21 it becomes 6 between (…) 28 and 36 it becomes 8 between and it stands above in the first row. (Class T, p. 4)

and

> On the third row stands 4 – 10 – 20 – 35 – and so on.
> Between 4 and 10 is it 6 between. (…) Between 56 and 84 is it 28 between, and it stands above as it did in the first row. (Class T, p. 4-5)

and

> In the fourth row is it exactly the same as in the first and second and third row. (…) They stands above, those numbers that were between. (Class T, p. 6)

and

> Far out to the edge stands 3 and next to stands 6 and above stands 3 and $3+3=6$.
> 6 and next to stands 10 and above 4 and $4+6=10$. (Class T, p. 10)

These quotations are all paraphrases over the same theme; a particular number is ‘between’ two other particular numbers. When the pupils are saying that a number is ‘between’ two other numbers they are expressing either the difference between those two numbers or what shall be added to the smallest of the two numbers in order to get the biggest of the two. In this context and using symbols it can be written:

$$W(h,v) - W(h-1,v) = W(h,v-1) \text{ or } W(h-1,v) + W(h,v-1) = W(h,v)$$

In any cases this is identical with (10.15). Mathematically this relationship is a recursive defined function in two variables.

At page 6 in their answer is found “What stands up the street in the second row stands down the street in the second row. The same happens up the street and down the street.”, which means that the symmetry property of the numbers in the table has been discovered. The pupils have here expressed the relationship $W(h,v) = W(v,h)$, which is identical with (10.2).

By using a starting value and applying the relationships (10.13), (10.2), and (10.15); i.e.
$W(1,1) = 2$, $W(h+1,1) = W(h,1)+1$, $W(h,v) = W(v,h)$, and $W(h,v) = W(h-1,v) + W(h,v-1)$ where *h* and *v* both are bigger than 1, Class T was able to complete the table. This was then done. Below is an example:

	1 hus bortover	2 hus bortover	3 hus bortover	4 hus bortover	5 hus bortover	6 hus bortover	7 hus
1 hus nedover	2	3	4	5	6	7	8
2 hus nedover	3	6	10	15	21	28	36
3 hus nedover	4	10	20	35	56	84	120
4 hus nedover	5	15	35	70	126	210	330
5 hus nedover	6	21	56	126	252	462	792
6 hus nedover	7	28	84	210	462	924	1716

Dei svarte sjekka vi med og teikne opp.
Dei blå har vi ikkje sjekka med å teikne men vi rekna dei ut.

Figure 10.4. Class T, page 8

The Norwegian text reads:

> The black ones we checked by drawing.
> The blue ones have we not checked by drawing but we calculated these.

The teacher confirms this interpretation, point 6 in his description:

> After the pupils had discovered the relationships between the numbers in the table, they were able to fill in the rest of the table. (Blue numbers)
> We took then a spot test with 4 houses up the street and 3 houses down the street, which after the table should give 35 shortest ways.
> The pupils tried first in groups of 4. Afterwards we jointly draw the shortest ways, and discovered that what we had calculated was correct. (Class T, p. 1)

All together eight such drawings, the jointly one and seven ones made by the pupils, were part of the answer. The jointly executed drawing:

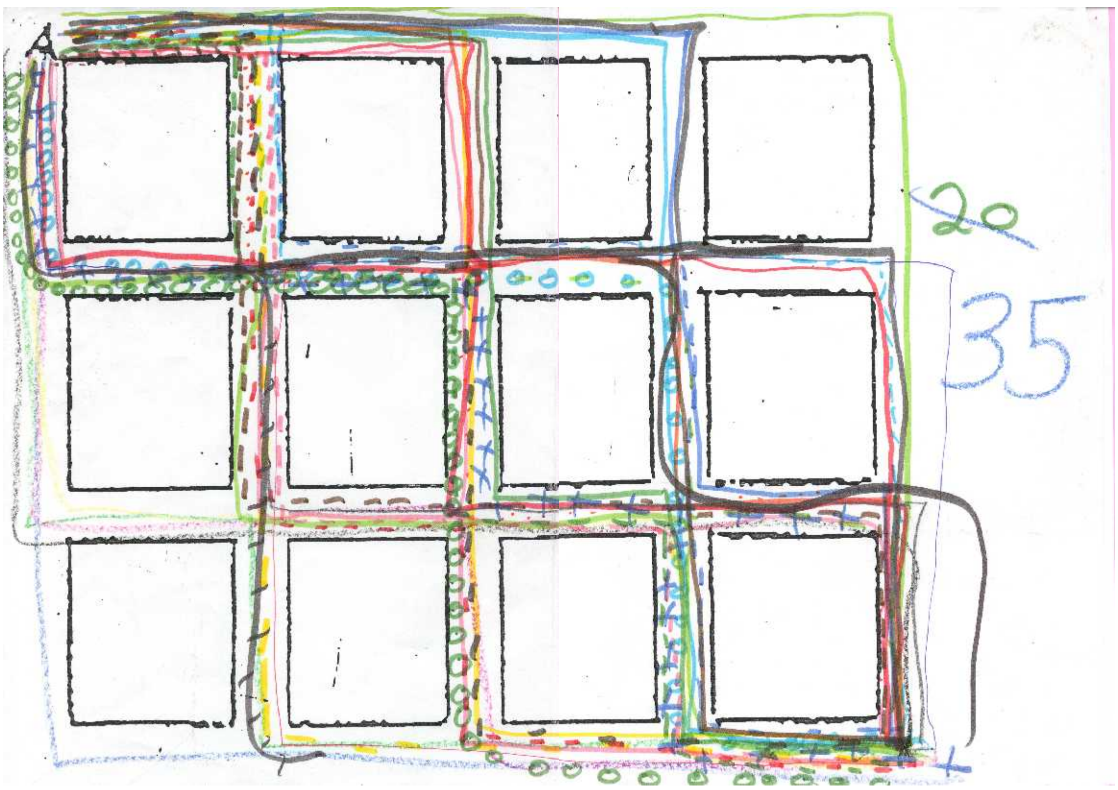


Figure 10.5. The final drawing of the ways from A to (4,3) (Class T, p. 9)

The original drawing is a A3 sheet, and indeed, it is possible to identify all the 35 shortest ways on the original.

At the end of this section Class T raise the following question:

> Our school is located 6 houses up the street and 6 houses down the street away from Anne. How many shortest ways can Anne go to the school? (Class T, p. 7)

The correct answer is 924, which is given by the class. See also the figure 10.4 at page 198.

10.4.2 The probability to arrive at *B*

This part of the pupils' solution consists of five tables and five written texts that may be called essays, which describe how the class worked and also their findings, and the solution of this part of the task. In addition is the teacher's detailed description of the working process to the class. His description of this particular part, point 7 in his list, is:

> I explained then for the pupils about the two different options, heads and tails, by tossing a coin. We did that concrete. We let one desk be one house up the street and one house down the street and noticed that we now and then came to the right and some times not to the right terminal point. (Class T, p. 1)

The pupils' have also described this activity, for example:

> Mohamad started by tossing a coin. When he got heads he went up the street and when he got tails he went down the street. (Class T, p. 13)

This indicates strongly that the class modelled the city in the classroom, and then 'performed the problem'. The performance revealed that they now and then, arrived at the right terminal-point, $B=(2,4)$, and sometimes they did not arrive at *B*. They had thus realised that it was possible to come to *B* applying the tossing a coin procedure, but they had still to figure out how often they came 'right'. The question faced was: How to continue in order to do the counting? Point 8 in the teachers list throw light on that question:

> We did then the same by attaching ropes around the waists to 6 pupils. Each of these 6 got in this manner a tail. The pupil in the start position held two tails; One that went in the heads-direction and one that went in the tails-direction. Each of these two then were holding two new tails. They observed that 2 pupils came to the right position and that 2 pupils came on wrong spots.
> The pupils understood now the principle… (Class T, p.2)

The principle introduced by the teacher: If the tossed coin showed heads they followed the rope, the tail, to the nearest pupil in the heads-direction, and correspondingly for the tails-direction. One tossing of a coin was equivalent to move along one of the two tails. This principle was firstly demonstrated on a small scale in their classroom, after finishing this demonstration they went to a bigger room. In this room the teacher draw on the floor, a map of the city and they repeated and also extended the activity from the classroom. Below is some descriptions of this activity given by the pupils.

> Firstly a pupil took (one's) stand on the start. Then two pupils got tails and white hats. Neither came on the right spot (position). Then came 4 more pupils 2 of 4 came on the right spot. They had pink hats. Then came 8 pupils they had green hats. Three of 8 came on right spots. (Class T, p. 14)

and

> We made tails of rope and tied it around the waist.
> One stood and held two tails. Each of us went his own way.

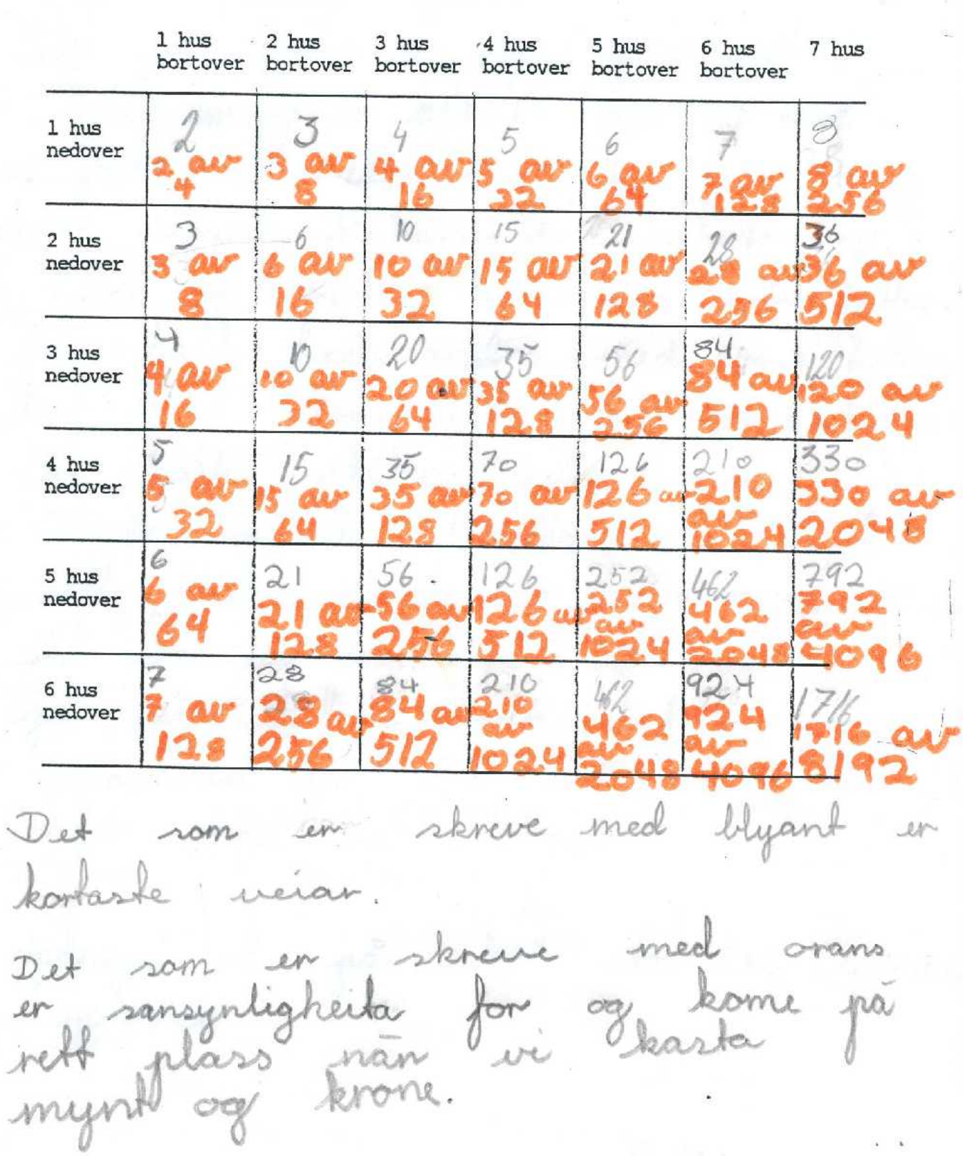
> We observed how many that came on the right spot. Some came right and some came wrong. (Class T, p. 10)

A drawing of this activity, made by one of the pupils, is:

Figure 10.6. An illustration of the activity performed by Class T, (p. 18)

Even though the different descriptions of this activity are relatively detailed the interpretation is not straight forward. They are writing about tails, hats of different colours, and that some of them 'came right or wrong'. From figure 10.6 it is observed that the pupil at the top is the only one without a hat, and without a tail around his waist. This pupil is in the *A* position. Further is it observed that pupils with hats of the same colour are located at equal distances from the pupil at position *A*; i.e. they could be classified as being on the same level. The pupils with the white hats are located at level one, those with pink hats on level two, the green one's at level three etc.. Each pupil can be associated with co-ordinates, (h, v), and for pupils at the same level the sum of *h* and *v* is identical; e.g. level one: $h + v = 1$, level two: $h + v = 2$ etc.. Related to Pascal's triangle the numbering of the levels is identical with the numbering of the lines in Pascal's triangle.

Then it is the question of 'coming right or wrong'. In addition to the information found in the written texts, the drawings Class T made of this activity give valuable information. Combining the information found in the texts and in the drawings reveals: The two pupils located at (1,0) and (0,1), those two with the white hats, were classified as 'came wrong'. As an example the four pupils with pink hats; the two located at (1,1) came to a spot that was classified as a right position, and those two at (2,0) and (0,2) came to 'wrong' spots. The three pupils with green hats located at either $(1,2)$ or $(2,1)$ were classified as coming to a spot characterised as a right position. It is noticed that all the spots classified as wrong positions are T-crosses, and such crosses were omitted by Class T in the first part of this task, see page 195. However, if this interpretation is correct or not is found of little or no importance for the further analysis of this solution. The main reason for this is that the problem Class T was facing was to keep order over how many persons at each level, and how many of those that came 'right', and what is equal important, how many that came 'wrong'. The importance is therefore not if one or several persons have been classified as being on a spot classified as a 'right' or a 'wrong' position, but how many persons on that spot, and how many persons on that level. In order to get a systematic overview of the number of persons on a spot and on that level tables was constructed. Below is a facsimile of one of these tables.



	1 hus bortover	2 hus bortover	3 hus bortover	4 hus bortover	5 hus bortover	6 hus bortover	7 hus
1 hus nedover	2 2 av 4	3 3 av 8	4 4 av 16	5 5 av 32	6 6 av 64	7 7 av 128	8 8 av 256
2 hus nedover	3 3 av 8	6 6 av 16	10 10 av 32	15 15 av 64	21 21 av 128	28 28 av 256	36 36 av 512
3 hus nedover	4 4 av 16	10 10 av 32	20 20 av 64	35 35 av 128	56 56 av 256	84 84 av 512	120 120 av 1024
4 hus nedover	5 5 av 32	15 15 av 64	35 35 av 128	70 70 av 256	126 126 av 512	210 210 av 1024	330 330 av 2048
5 hus nedover	6 6 av 64	21 21 av 128	56 56 av 256	126 126 av 512	252 252 av 1024	462 462 av 2048	792 792 av 4096
6 hus nedover	7 7 av 128	28 28 av 256	84 84 av 512	210 210 av 1024	462 462 av 2048	924 924 av 4096	1716 1716 av 8192

Det som er skreve med blyant er kortaste veiar.

Det som er skreve med orans er sansynligheita for og kome på rett plass når vi kasta mynt og krone.

Figure 10.8 One of the probability tables to Class T (p. 16)

The hand written text reads:

What has been written with pencil is shortest ways.
What has been written with orange is the probability to arrive at the right spot when we tossed heads and tails.

Figure 10.9gives a strong indication that Class T realised that the importance was, for each of the positions in the city, to determine two numbers. One number that stated the number of persons on that particular spot, and one that kept order of the number of persons with hats that had identical colour; i.e. the number of persons at that level. The main reason for asserting this is that in each cell in this table there has been written, in orange, '*a* of *b*', where *a* and *b* are numbers, and the text under the table explains: "What has been written in orange is the probability for coming to the right spot when we tossed heads and tails." (Class T, p. 16). Thus to each position there is associated two numbers, and these numbers Class T expressed as '*a* of *b*', where *a* is the number of persons on that position and *b* is the number of persons on that level; e.g. position $(1,0)$: 1 of 2; position $(1,1)$: 2 of 4, etc.. By means of this activity the pupils were able to determine the number of persons on the spots nearest to *A*, and also the number of persons on the lowest levels:

> Mohammad[12] started first. Then came two with white hats after that came four with pink hats after that came eight with green hats and after that came sixteen with red hats. (Class T, p. 11)

Using the notation from section 10.2.3 the findings to Class T can be symbolised as:

Level one: $p(1,0) = p(0,1) = \text{'1 of 2'}$.

Level two: $p(2,0) = p(0,2) = \text{'1 of 4'}$, and $p(1,1) = \text{'2 of 4'}$.

Level three: $p(3,0) = p(0,3) = \text{'1 of 8'}$, and $p(1,2) = p(2,1) = \text{'3 of 8'}$.

Level four: $p(4,0) = p(0,4) = \text{'1 of 6'}$, $p(1,3) = p(3,1) = \text{'4 of 16'}$, and $p(2,2) = \text{'6 of 16'}$.

It is noticed that Class T has expressed these probabilities as ratios, and not as fractions or decimal numbers. Expecting Class T, a grade 3 class, to use fractions and/or decimal numbers in stating and communicating these probabilities is found to be of little realism. The main reason for that is found in the curriculum. According to M87, the curriculum at that time, fractions and/or decimal numbers were introduced in grades 4 to 6. For grades 1 to 3 only "Examples of the most common fractions such as a half, one third, a quarter" should be introduced ((The) Ministry of Education and Research, 1990, p. 213). In addition, in everyday life ratios is a frequently used method of stating and communicating probability. In order to go on with this activity they had a shortage of persons, but then they discovered:

[12] This time the name was spelled with a double m.

We intended to go and get more pupils. But then we discovered that was not necessary to do, because we found a way to calculate it. It was only to calculate in the same manner as we did when we calculated with ways. The table became like 8 + 8 = 16 it was standing like

	8
8	16

The number of those that came right became like

	3
3	6

just as the with the ways we coloured with crayons. (Class T, p. 14-15)

The pupils edited the last part of this quotation as:

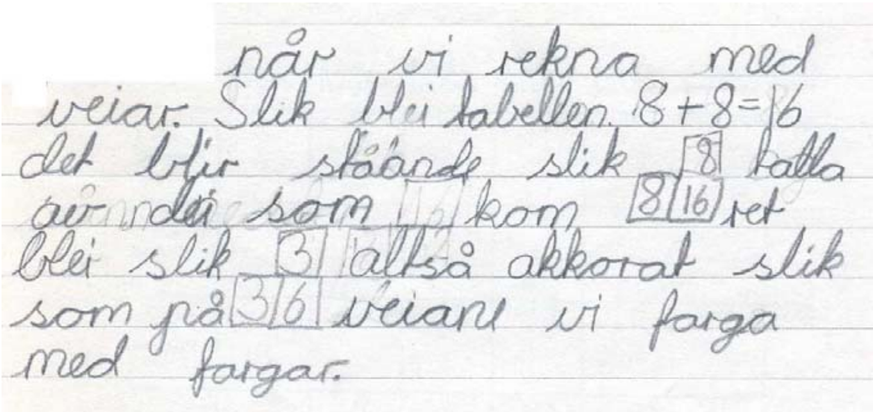


Figure 10.7. The way Class T edited their explanation (p. 15)

Two relationships can be identified in the above quotation. The first one stated how the number of persons at a fixed level can be calculated, and this is described by: “It was only to calculate in the same manner as we did when we calculated with ways.” (Class T, p. 14-15). This strongly indicates that the intention with the given number example, $8+8=16$, was to exemplify this relationship. The second relationship focuses on how to calculate the number of persons at a fixed or ‘right’ position. They discovered that this was identical with the relationship found in the first part of this task, (10.15); i.e. the number of persons at a fixed position was identical with the number of shortest ways from *A* to the actual position.

The relationship between a fixed position, (h,v), and the level *t* to which this position belongs is, as mentioned at page 200, given as $t=h+v$. Calling the number of persons at level $h+v$ for $l(h,v)$, the first relationship, how to calculate the number of persons at a fixed level, can be symbolised by:

$$l(h,v)=l(h-1,v)+l(h,v-1) \qquad (10.16)$$

Again it is observed that Class T has stated a recursive relationship involving a function in two variables. Class T has not explicitly stated starting values, but they knew from the activity that

$$l(1,1)=4\ ,\ l(1,2)=l(2,1)=8 \text{ and } l(1,3)=l(2,2)=l(3,1)=16 \qquad (10.17)$$

Class T observed "…that diagonally stood the same numbers." (Class T, p. 12). The question is which 'same numbers' are Class T referring to? The only numbers in their table that satisfy this requirement is the number of persons at each level. This can be symbolised as:

$$l(1,v)=l(2,v-1)=l(3,v-2)=\ldots=l(h-1,2)=l(h,1) \qquad (10.18)$$

They were also most probably aware of the fact that the sum of those that came 'right' and those that came 'wrong' at one of the first four levels is identical with number of persons at that level, which is relationship (10.4), see page 191. (At page 200 it is argued that they realised from the activity that at each level it was always two persons that came 'wrong'. One person down the street and one up the street; i.e. those at positions $(0,v)$ and $(h,0)$.) By using the relationships (10.16), (10.17) and (10.18) $l(h,v)$ can be calculated for all $h>0$ and $v>0$. The result of this calculation is that $l(h,v)$ is a power of 2, or to be more precisely 2^{h+v}. The correctness of this result follows directly from (10.4). According to *M87* powers should be introduced in the grades 7-9 ((The) Ministry of Education and Research, 1990, p. 214), so it is quite natural that Class T did not express $l(h,v)$ as powers of 2. From the starting values 2, 4, 8, and 16 they continued by doubling at least to $8192=2^{13}$, see facsimile page 201. Since Class T discovered that the number of persons at each position was identical with the number of ways from *A* to that position, see page 203, the calculation of that number was not longer a problem; it was identical with the calculation done in the first part of this task. The probability to arrive an arbitrary position (h,v), where $h>0$ and $v>0$ can then be symbolised as

$$p(h,v)=\text{'}W(h,v) \text{ of } l(h,v)\text{'} \qquad (10.19)$$

and since both $W(h,v)$ and $l(h,v)$ can be calculated, $p(h,v)$ is determined. Mathematically, $p(h,v)$ is a ratio between two functions in two variables. To determine the probability for Anne to arrive at $B=(2,4)$, by tossing a coin, was as then an easy task for Class T "The probability that Anne arrives at Berit's place is 15 of 64." (Class T, p.17).

If the operation of adding the probabilities, in the way Class T has done, is symbolised by $\oplus$ the calculation of $p(h,v)$ can be written as:

$$
\begin{aligned}
p(h,v) &= p(h-1,v) \oplus p(h,v-1) \\
&= 'W(h-1,v) \text{ of } \ l(h-1,v)' \oplus 'W(h,v-1) \text{ of } \ l(h,v-1)' \\
&= '(W(h-1,v)+W(h,v-1))' \text{ of } \ '(l(h-1,v)+l(h,v-1))' \\
&= 'W(h,v) \text{ of } \ l(h,v)'
\end{aligned}
\tag{10.20}
$$

The calculation of the probabilities (10.20) shows that Class T used what may be called a variant of the recursive relationship (10.12). As commented on at page 202 Class T used ratios when they stated a probability. These ratios can be identified with fractions, the relationship (10.19) can then be written:

$$
p(h,v) = \frac{W(h,v)}{l(h,v)} \tag{10.21}
$$

Using this form the calculations (10.20) can be written:

$$
\begin{aligned}
p(h,v) &= p(h-1,v) \oplus p(h,v-1) \\
&= \frac{W(h-1,v)}{l(h-1,v)} \oplus \frac{W(h,v-1)}{l(h,v-1)} \\
&= \frac{W(h-1,v)+W(h,v-1)}{l(h-1,v)+l(h,v-1)} \\
&= \frac{W(h,v)}{l(h,v)}
\end{aligned}
$$

10.4.3 Summary – Class T

The teacher to this class structured the task more detailed than in the original text. He also arranged for the activities executed by the pupils. The class worked on this task over a period of three weeks and this lead to a very comprehensive document. It is argued that mathematically their investigations lead to several recursive relationships defined by means of functions in one and two independent variables. Function defined as a ratio between two functions in two independent variables is also applied by this class.

Mathematical structures inherent in this solving procedure:

- Recursive defined functions in one and two independent variables;
- Function defined as a ratio between two functions in two independent variables.

10.5 Class U

10.5.1 The number of ways

The pupils made a model of a mini-city consisting of 5×5 houses. In order to explore how many different ways from *A* to some of the nearest crosses to *A*, threads with different colours were placed in the model. They concluded that there were 15 different threads between *A* and *B* that had equal length (Class U, p. 1-2). Correspondingly the number of different ways between *A* and *C* was found to be 10. This last result was illustrated:

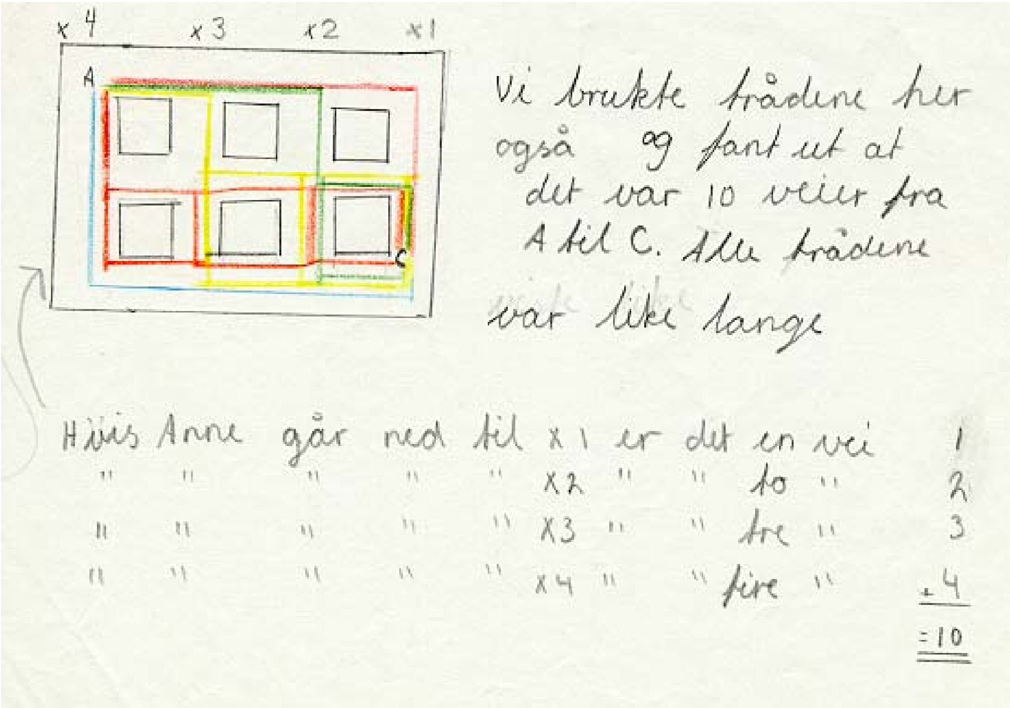


Figure 10.9 Illustration of the different ways between *A* and *C* (Class U, p. 2)

The Norwegian text on the right side of the drawing reads:

> We used the threads also here and discovered that it was 10 ways from A to C. All the threads were of equal length.

The text beneath the drawing reads:

If	Anne	goes	down	to	×1	is	it	one	way	1
"	"	"	"	"	×2	"	"	two	"	2
"	"	"	"	"	×3	"	"	three	"	3
"	"	"	"	"	×4	"	"	four	"	+ 4
										= 10

On the drawing, figure 10.9, is the way that goes through the lower left corner, position $(0,2)$, drawn twice, one with the blue colour and one with orange colour, however it has only been counted once. The organising of the counting is found to be interesting and will be commented at page 209.

As was the case for Class T, Class U did not state that to each T-cross there is exactly one way. Most probably this is found to be obvious, and therefore of little or no interest to state. For the remaining part of this section and if nothing else is stressed, a cross is an inner cross; i.e. if the position is (h,v) then $h \cdot v \neq 0$.

In order to investigate the number of ways to the crosses nearest to *A*, they continued:

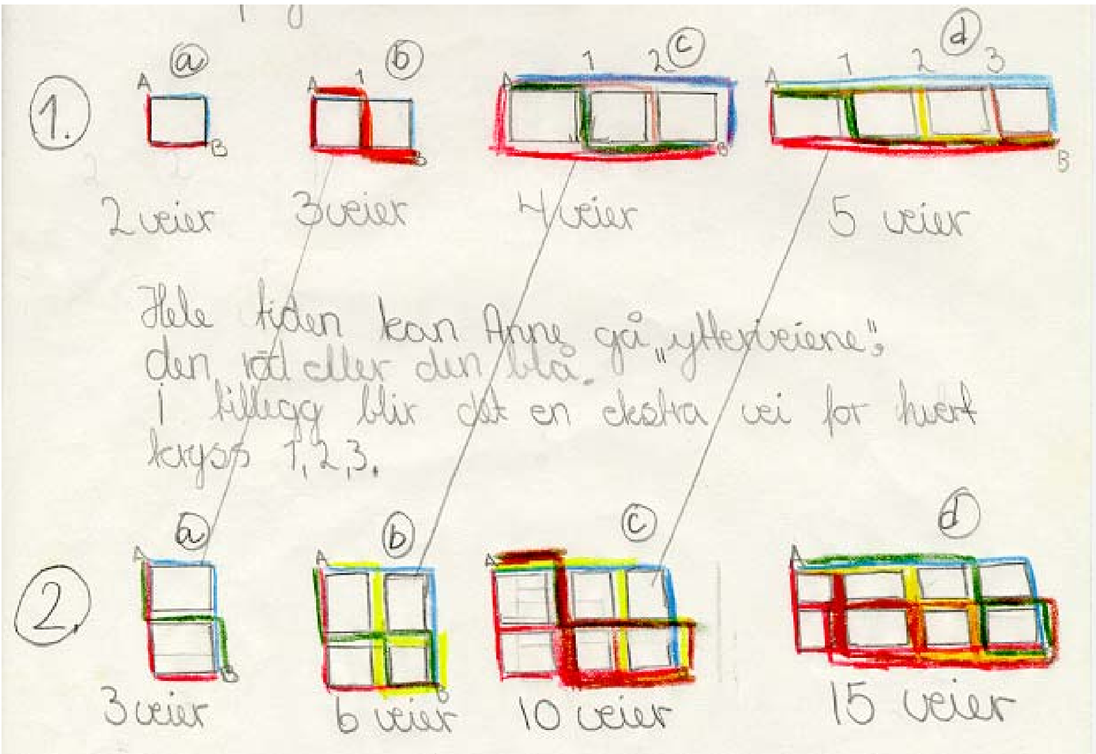


Figure 10.10 Illustration of the different ways to the crosses nearest to *A* (Class U, p.3)

The Norwegian text reads:

> All the time Anne can walk the "outer ways",
> the red one or the blue one.
> In addition is it one extra way for each cross 1, 2, 3.

The first part of figure 10.10, point 1., establish the number of ways to crosses with co-ordinate $(h,1)$, and they discovered that

$$W(1,1) = 2,\ W(2,1) = 3,\ W(3,1) = 4, \text{ and } W(4,1) = 5.$$

The text that follows directly after the first four figures, 1.a, 1.b, 1.c, and 1.d, argues for the correctness of these numbers. The argumentation is not based on counting, but on relationships found in the figures. Their argumentation can be separated in two parts:

- The ‘outer ways’ are two, the red and the blue.
- There is one extra way for each cross 1, 2, 3.

The interpretation of the first of these two causes no difficulties. To each position there are at least the two different ways, the ‘outer ways’. The second part emphasises that the number of ways varies according to a certain pattern when the position to a cross changes. If an extra cross is added then the number of ways increases by one. For example to position $(1,1)$ there are only the two ‘outer ways’, to position $(2,1)$ there is in addition to the two ‘outer ways’ one ‘zigzag’ way. The relationship inherent in their argumentation can be described by:

$$W(h,1) = 2 + 1 \cdot n \tag{10.22}$$

where the number *n* is dependant of the number of crosses; i.e. the position $(h,1)$. Their drawing reveals that a certain kind of crosses plays an important role in determining *n*. From figure 10.9 it is observed that Class U has counted or numbered the T-crosses between *A*, $(0,0)$, and $(h,0)$, and that they are referring to those crosses in the text. This strongly indicates that those crosses play a crucial role in determining *n*. The interpretation of the statement “In addition is it one extra way for each cross 1, 2, 3.” (Class U, p.3) is most likely that *n* equals the number of T-crosses between $(0,0)$ and $(h,0)$. The number of such crosses is $h-1$, hence $n = h-1$. The relationship (10.22) can then be transformed to:

$$W(h,1) = 2 + 1 \cdot (h-1) = h+1 \tag{10.23}$$

Class U has not stated that the relationship (10.23) is valid for all *h*. Their city, or their model of the city, consisted, as mentioned at page 206, of 5×5 blocks, but the argumentation done by the class is valid for all crosses with co-ordinate $(h,1)$. The relationship (10.23) is an explicit function in one independent variable.

The four illustrations 2.a, 2.b, 2.c, and 2.d in figure 10.10 establish the number of ways to the crosses with co-ordinates $(h,2)$, $h = 1,2,3,4$. For these numbers the stated relationship can be symbolised:

$$W(h,2) = W(h-1,2) + W(h,1) \tag{10.24}$$

This relationship was described as:

> We found out that the ways in 1.b and the ways in 2.a together become the ways in 2.b
> $\underline{3+3=6}$
>
> We noticed also that the ways in 1.c + 2.b together become the ways in 2.c
> $\underline{4 + 6 = 10}$ (Class U, p. 3)

The lines drawn with pencil in figure 10.10, for example the one between 1.b and 2.a, indicates or stresses relationship (10.24) for $h = 2$. In order to verify if this relationship also was valid for other positions in their city, they decided to do a test. They went on with the explorations in the same manner as done before. The result of these explorations, written at page 4, can be symbolised:

$W(1,3)=4$, which they named 3a;
$W(2,3)=10$, named 3b;
$W(3,3)=20$, named 3c;
and $W(4,3)=35$, named 3d.

They continued:

> If the relationship we discovered is correct also now, then the ways in 3a + 2b shall be the ways in 3b that is correct since 4 + 6 = 10 (Class U, p. 4)

As seen from the above quotation they concluded that a relationship similar to (10.24) was valid. At page 7 in their solution Class U presented the following table:

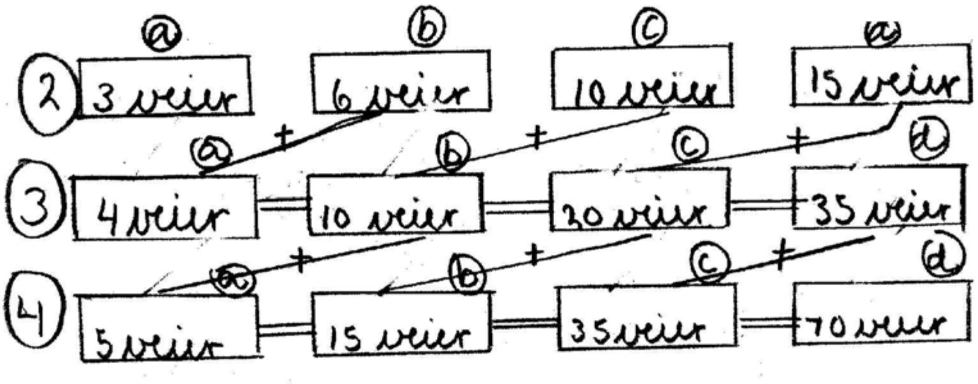


Figure 10.11 The number of different ways to some of the crosses in the city (Class U, p. 7)

This figure gives a very strong indication that they discovered or realised that the relationship (10.24) could be extended to the all positions (h,v) in their city, where $h>1$ and $v>1$; i.e. for $h>1$ and $v>1$ the number of ways was given by the relationship:

$$W(h,v)=W(h-1,v)+W(h,v-1) \tag{10.25}$$

As is well known the relationship (10.25) is correct. Mathematically it is a recursive defined function in two independent variables.

10.5.2 Relationships

At page 4 in this solution starts a section with the heading “We discovered also something strange”. This section concentrates on the relationship Class U described and applied when they counted and checked the number of ways from *A* to *B*, see for example figure 10.9 page 206. In order to be convinced that all the ways have been counted exactly once it is necessary to elaborate a systematic counting process or counting strategy. Form figure 10.9 it appears that Class U introduced two counting aids. Firstly, Class U identified four of the T-crosses with separate symbols, $\times 4=(0,0)$, $\times 3=(1,0)$, $\times 2=(2,0)$, and $\times 1=(3,0)$. Secondly the coloured threads formed a special pattern. From what they have written beneath the drawing in figure 10.9 the counting strategy used can be described by:

> Firstly count all the ways that goes through $\times 1 = (3,0)$, the read thread;
> then those that goes through $\times 2 = (2,0)$, and not through $\times 1 = (3,0)$, the green threads;
> then those that goes through $\times 3 = (1,0)$ and not through $\times 2 = (2,0)$, the yellow threads;
> and lastly those that goes through $\times 4 = (0,0)$, and not through $\times 3 = (1,0)$, the orange threads.

As seen on the pages 4-7 in Class U's solution, a similar counting strategy has been applied in many other cases. Firstly the horizontally T -crosses have been identified by a symbol, $\times t$, where t is a number. The T-cross furthest away from A was identified as ×1, then the T-cross to the left of this one as ×2, etc. ending at A. The symbol that identified A was then dependant of the number of columns of houses. If the number of columns was h, then A was symbolised as $\times(h+1)$. For example with three columns of houses A was named ×4, if there was 2 columns A was ×3. Class U did not explicitly state that the number of ways to position (h, v) via $\times 1 = (h, 0)$ is one, but from their solution it appears that they have realised this fact. Using the symbols introduced by Class U, the counting strategy in figure 10.9can be symbolised by:

$$2\text{c} = 1 + 1\text{a} + 1\text{b} + 1\text{c} \tag{10.26}$$

or by using the symbol from section 10.2, the relationship (10.26) can be written:

$$W(3,2) = 1 + W(1,1) + W(2,1) + W(3,1) \tag{10.27}$$

Relationship (10.27) is a number example of relationship (10.3), page 191, The Christmas Stocking Theorem.

In addition to the example found in figure10.9, Class U has given seven other number examples of relationship (10.27). The last one of these examples states the number of ways to position $(4,4)$:

If	Anne	goes	to	×1	is	it	one	way	1
"	"	"	"	×2	"	"	four	"	4
"	"	"	"	×3	"	"	ten	"	10
"	"	"	"	×4	"	"	twenty	"	20
"	"	"	"	×5	"	"	thirty-five	"	+ 35
									= 70

(Class U, p. 7)

The reason for why they ended with this example is most likely that in their model of the city position $(4,4)$ is the position furthest off from A. The above example can be symbolised:

$$W(4,4) = 1 + W(1,3) + W(2,3) + W(3,3) + W(4,3) \tag{10.28}$$

Generally, these seven relationships given by Class U can be symbolised:

$$W(h,v)=1+W(1,v-1)+W(2,v-1)+\ldots+W(h,v-1) \quad (10.29)$$

where $0<h<5$ and $1<v<5$.

These relationships are also present in the table that ends this part of the solution, see figure 10.11. The way this table is presented indicates strongly that Class U discovered that: If one to the number one adds, from left, one or more of the numbers in a row, the sum is the number in the next row which is located under the last added number.

The two first lines in figure 10.11 are a summary of what Class U has written before the 'something strange' section. Most probably this means that some of the numbers in these two lines have been calculated by relationship (10.25) and, in one way or another, controlled by counting the coloured threads, or vice versa, first the counting procedure and then it was controlled that (10.25) was valid. It is found quite likely that the numbers in the last line in figure 10.10 have been calculated by (10.25), but it is also possible that some of these numbers have been calculated by relationship (10.29) and that (10.25) has been used as a control.

10.5.3 Summary – Class U

Based on a well organised counting procedure the number of different ways to the nearest crosses to *A* was stated. Investigation of these numbers resulted in several relationships. It is argued that in their solving procedure Class U applied an explicit function in one independent variable, and recursive defined functions in two independent variables.

Mathematical structures inherent in this solving procedure:

- Explicit defined function in one independent variable;
- Recursive defined functions in two independent variables.

10.6 Class V

This was the second time Class V participated in the *Tangenten* competition. Their first participation was, as Class P, with a solution to the task *Black and White Squares*, see section 8.7. Attached to the pupils' solution was a letter from the teacher. In this letter it was for among others given some information about both the working processes to the class.

10.6.1 The number of ways

A vital part of this solution is a relatively comprehensive table constructed by means of a spreadsheet, (Class V, p. 4). This table gives a survey over the number of ways between *A* and all positions (h,v) where $0\le h\le 8$ and $0\le v\le 40$. The table was organised as table 10.1.

The construction of the spreadsheet is explained in the solution.

> Firstly we wrote all the ones' onwards and downwards. Then we wrote that the number in first column second row added with the number in second column first row equals the number in second column second row. After that we copied to the whole spreadsheet. (Class V, p. 2)

As seen, this explanation is unambiguous, and it can be symbolised:

$$
\begin{aligned}
&W(h,0) = W(0,v) = 1 \text{ where } h \geq 0 \text{ and } v \geq 0 \\
&W(h,v) = W(h-1,v) + W(h,v-1) \text{ if } h > 0 \text{ and } v > 0
\end{aligned}
\tag{10.30}
$$

Relationship (10.30) is recursive, and it is identical with (10.1). Mathematically (10.30) is a recursively defined function in two independent variables. The constructed table has, as mentioned above, upper bounds for both *h* and *v*. The way Class V has explained their construction of the spreadsheet gives no indication for the necessity of limiting *h* and *v*. The limitation is most likely due to practical causes, the size of an A4 sheet.

It is noticed that Class V has given an argument for why the number of ways to *A* is one, "A is the one in the top left-hand corner because the only possibility to arrive there is to stand still." (Class V, p. 4). They did not argue for why (10.30) gives the number of ways to the other positions. Why they did not argue for the correctness of (10.30) is impossible to answer clearly in retrospect, but it could be that they took it for granted or consider it to be unnecessary since it was not directly asked for in the text.

10.6.2 Pascal's triangle

As known from mathematics constructing Pascal's triangle using a recursive relationship as (10.1) or (10.6) is straight forward, and as experienced from teaching the opposite is neither nor a difficult task; i.e. on the basis of the number pattern in Pascal's triangle deduce the recursive relationship defining this number pattern. However, changing the focus from a recursive to an explicit perspective complicates these two processes, the construction and/or the deducing. The construction of Pascal's triangle by using an explicit relationship as (10.7) is not difficult, but it is certainly more awkward. Deducing an explicit relationship as for example (10.7) on the basis of the number pattern in Pascal's triangle is, as mentioned at page 190, quite another question, and is not straight forward. Based on experiences from own teaching practices, only in a few exceptional cases students have been able to discover such a relationship directly from the number pattern. However, Class V discovered or stated the explicit relationship, but this discovery was, as seen from the letter to the teacher, not based on a direct observation of the number pattern.

> One pupil has been in charge of more than the half of the results. He is one of the two most gifted pupils in the class. (…) He has also once with a little starting assistance found the chance for complete house in Lotto. After some weeks he said, "This is the same". (Class V, p. 1)

This particular pupil discovered after some weeks grappling that the problem in question was similar, or actually identically with a problem he had solved before, determining the number of series in *Lotto*; i.e. to draw an unordered sample of 7 numbers out of 34. This progress is a nice demonstration of one of the heuristic strategies in problem solving emphasised by *Polya* (1957). In his letter the teacher also stressed that this pupil had the knowledge, it took just some time to activate it (Class V, p. 1). On the basis of this discovery the class, or to be more correct, the pupil was able to construct or deduce an explicit relationship for the numbers in Pascal's triangle. The analogy to the Lotto problem told this particular pupil that out of a certain set of objects, in this case numbers, should be drawn, randomly, some objects, a sub set, and the number of such sub sets was the interesting number. Knowing the number of objects and the number of the randomly drawn objects, he knew also how to calculate this interesting number, the number of sub sets.

Using the symbols from section 10.2.2, the way Class V presented the explicit relationship can be characterised by: Firstly is explained how to determine the numbers *n* and *r* in the binomial coefficient $\binom{n}{r} = 15504$, then it was demonstrated how to use these two numbers in a calculation ending with the answer 15504 (Class V, p. 2). Even though the presentation was based on a particular position in the Pascal triangle, the explanation had a general character. It can be described by:

> From the actual position in the table, move along the diagonal to the left, stop with the number in the column just before the column consisting of ones. The number stopped with is *n*.
> The original position in the table is found in column *r*, where the numbering of the columns starts with the first column after the columns of ones.

By using this method they determined the values *n* and *r*, for the chosen position, to be $n = 20$ and $r = 5$. Next was the calculation, which they know was to calculate a rational quantity where both the numerator and the denominator were products of $r = 5$ consecutive natural numbers. The biggest factor in the product in the numerator should be $n = 20$, and the denominator was the product of the first $r = 5$ natural numbers. Hence the number of ways could be calculated as $\frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 15505$, which is correct.

The procedure given by Class V for determining both *n* and *r* and then calculate the number of ways, is correct also for an arbitrary position in the city. If the position is (h, v) then, linked to the co-ordinate in Pascal's triangle, the procedure used implies that $n = v + h$ and $h = r$. The number of ways can then be symbolised as

$$W(h, v) = \binom{n}{r} = \binom{v+h}{h} \qquad (10.31)$$

which is an explicit defined function in two variables that is identical with (10.5).

10.6.3 Other relationships

> If we multiply the number in the second column and the first row with the number in second column and third row and then divide the product by two we then get the number of shortest ways from A to the cross which is lying two blocks up the street and one block down the street from A. Like that can we do for each two consecutive numbers in the second column in the spreadsheet, in order to find the number that is in the third column and the same row as the upper of the two numbers we multiplied. (Class V, p. 2)

There is a mismatch in the first sentence and also between the first half and the second half of this quotation. Most likely this mismatch is just a typing error. The number in the position referred to as "second column and first row" is one, and "the number in second column and third row" is three, and their product divided by two is not two, which it had to be if the description was correct. If the position of one of these two numbers is changed to 'second column and second row' then there is no mismatch. The term 'second column' refers to the second column in their table, that is the column where $h=1$, similarly for the 'third column' $h=2$. With mathematical symbols the relationship stated above can be written $W(2,v)=\frac{W(1,v)\cdot W(1,v+1)}{2}$, which is identical with (10.9). This relationship is a recursive defined function in one independent variable.

Class V has observed also that if all the number on a diagonal is added up, then the sum is a power of two. This relationship was formulated:

> If we add the number of different shortest ways from A to all the crosses that the diagonal passes will we find this system: 2, 4, 8, 16, 32, 64, 128 etc. The sum doubles for each block we go onwards or downwards. (Class V, p. 3)

The first part of this statement is in a way a weak form of (10.4) since it not links the numbering of the diagonals with the power of two. Calling the sum of the numbers in a diagonal for *S* and the diagonal for *d*, then the last statement in the above quotation can be symbolised

$$S(d+1)=2\cdot S(d) \qquad (10.32)$$

which is a recursively defined function in one variable.

10.6.4 Probability

The handling of the probability problem demonstrated another aspect of the problem solving process described by Polya (1957). The problem was separated into sub-problems, which was solved and then these solutions were combined to a solution of the original problem. Firstly Class V solved an analogous, but a simpler problem: What is the probability to arrive at $(h,0)$ or the equivalent problem: What is the probability to arrive at $(0,v)$? Their solution to the simpler problem was:

> The chance for coming from A to the first cross alongside our city when we toss heads and tails is 50%. For each cross alongside the chances is halved. (Class V, p. 3)

Using the symbols from section 10.2.3, the first sentence in the quotation above can be written:

$$p(1,0) = p(1,0) = 50\% \qquad (10.33)$$

The last sentence is a statement of calculating the probability for arriving at an arbitrary T-cross, tossing a coin. For the actual position, either $(6,0)$ or $(0,6)$, Class V calculated the probability to be 1,5625%, which is correct. The answer to the question of how Class V carried out this calculation, is, however, not found to be obvious. The main question in that respect is weather the calculation based on a recursive procedure or an explicit one. Or reformulated in a mathematical symbolic language, does the last sentence in the quotation indicate that they used relationships like

$$p(t+1,0) = p(0,t+1) = \frac{p(t,0)}{2} = \frac{p(0,t)}{2} \qquad (10.34)$$

or like

$$p(t,0) = p(0,t) = \frac{1}{2^t} = \frac{100}{2^t}(\%) \qquad (10.35)$$

It is found most likely that Class V has applied the recursive relationship (10.34). The main reason for that is that firstly they stated the probability to arrive at either $(1,0)$ or $(0,1)$ to be 50%, relationship (10.33). Then the next sentence states that the probability is halved for each T-cross. In this context this means that the probability to arrive at the next T-cross, either $(2,0)$ or $(0,2)$, is 25% and so on. The probability for arriving at one T-cross is half of the probability for arriving at the former T-cross at the same edge. Calculating the actual probability is therefore just to use formula (10.34) five times, where the starting value is given by (10.33). There is, however, a possibility that Class V has used a relationship like (10.35). Since there is six T-crosses from *A* to $(6,0)$ or $(0,6)$, *A* inclusive, is it necessary to carry through the halving process six times, or five times if the start position is either $(1,0)$ or $(0,1)$. That means the probability given by (10.33) has to be halved five times, or calculated as $\frac{50}{2\cdot 2\cdot 2\cdot 2\cdot 2} = \frac{50}{2^5}(\%)$.

Regardless of which method Class V used in their calculation of either $p(6,0)$ or $p(0,6)$, they calculated the probability for arriving at positions $(6,0)$ or $(0,6)$, which meant they had solved the simpler problem. The original problem remained. They know, however, just more than the probability for arriving at either $(6,0)$ or $(0,6)$, they know the probability for going one particular way to a block located at the sixth diagonal. Since there were 15 different ways from *A* to *B*, known from the first part of the solution, the probability for arriving at $B=(2,4)$ was therefore given as "$1,6525*15=23,4375\%$" (Class V, p. 3). Their method can be symbolised by:

$$\begin{aligned} p(2,4) &= p(6,0)\cdot W(2,4) \\ &= p(0,6)\cdot W(2,4) \end{aligned} \tag{10.36}$$

They continued "Like that can we calculate the chance for going to any cross in the city by tossing heads/tails." (Class V, p. 3). This indicates that Class V realised that the probability for arriving at an arbitrary position by tossing a coin can be symbolised as:

$$\begin{aligned} p(h,v) &= p(h+v,0)\cdot W(h,v) \\ &= p(0,h+v)\cdot W(h,v) \end{aligned} \tag{10.37}$$

where $p(h+v,0)$ and/or $p(0,h+v)$ are calculated by (10.34) or (10.35), and $W(h,v)$ by (10.30). Mathematically, relationship (10.37) is a product of two recursively defined functions in two independent variables.

10.6.5 Summary – Class V

This was a class with few pupils, and a large part of this solution was generated by one pupil with a particular interest in problem solving. Their investigations lead to explicit as well as recursive relationships, which in a mathematical language can be classified as functions. It is argued that there are recursive relationships involving one variable as well as relationships involving two variables.

Mathematical structures inherent in this solving procedure:

- Recursive defined functions in one and two variables;
- Explicit defined function in two variables.

10.7 Class W

Compared with the three former solutions to this task, this solution was very short, and it gave only the answers for the number of ways to *B* and *C*. For these two numbers Class W asserted that $W(B)=12$ and $W(C)=10$. The first of these is, as known from section 10.2.1, wrong, the second is correct. Both answers have been illustrated, drawings indicates the different ways between *A* and respective *B* and *C*. In the first of these illustrations three of the ways are missing, in the second all the ways have been plotted. In addition to the drawings Class W also argued for the correctness of these two values. In both cases they argued with basis in the concrete situation, but they gave also a general rule (Class W, p. 3), which can be formalised as:

In order to arrive at position (h,v) one have to chose direction in $h+v$ crosses, and since, in each of these crosses, there are two alternatives for this choice, the total number of ways is $2(h+v)$; i.e. $W(h,v)=2(h+v)$.

There is no mismatch between this argument and the drawings, but as known this argument is not valid for this particular task.

It is noticed that the rule stated by Class W is an explicit rule.

10.7.1 Summary – Class W

Mathematical structure inherent in the solving procedure:

- Explicit defined function in one variable.

10.8 Summary – In the city

Construction and investigation of the Pascal's triangle was the theme for this task. As is well known several relationships can be identified in this number pattern. One of the most visible relationships is the recursive formula that can be applied in the construction process. All classes, except one, discovered this relationship, which mathematically is a recursive function in two variables. The explicit formula that defines the values in Pascal's triangle was discovered or stated by one class, and one class stated an incorrect explicit formula. One class discovered the Christmas Stocking Theorem, another class the relationship about the sum of the numbers in a line in the triangle.

The probability question was answered by two of the classes. The grade 3 class stated this value as a ratio between two recursive defined functions, the other class, the grade 8 class, as a product between two functions.

An overview of the mathematical structures inherent in the solution procedures to the classes is given in the following table:

Class / Mathematical structure	T	U	V	W
Recursive defined function	X	X	X	
Explicit defined functions		X	X	X
Function defined as a ratio	X			

Table 10.2 Overview of the mathematical structures inherent in the solution procedures to the task *In the city*

11. Discussion

11.1 Introduction

The previous six chapters, chapter 5 to 10, focused on analysing the solution procedures applied by the pupils. The objective of this analysis was to uncover the mathematical structures inherent in these procedures. As explained in chapter 4 the analytical instrument applied for this purpose was a decoding of the answers into a mathematical language. This chapter discusses and reflects on the findings emanating from the study. As stated in chapter 1 the study addresses the following two research questions:

> ***What kind of mathematical structures are inherent in the solution procedures that pupils have applied in some of the open-ended tasks used in the Tangenten competition?***

and

> ***Is mathematical archaeology applied on pupils' written solutions of open-ended tasks a suitable tool for increasing the knowledge about pupils' mathematical activity and their mathematical thinking?***

11.1.1 Findings research question 1: Mathematical structures inherent in the solution procedures

As explained (elucidated) in chapter 3 the tasks used in this study were investigations, which in this context means that they have a closed starting situation and an open goal situation; i.e. an open ending. The structure of the tasks was more or less the same. A context that could be pure mathematical or to a certain extent an everyday occurrence that could be mathematised, was given. A appropriate description of this last context can be the same as Blum (1993) used to characterise one type of problems or situations applied in mathematical modelling, "an artificial dressing-up of a mathematical problem" (p. 4). In some instances the context was illuminated with an example. From this starting point the pupils were invited to carry through investigations. Based on these investigations they were challenged to formulate relationships or state one or more rules. For all the tasks, what may be called a *guided path* conducted the pupils in their investigational activity. One purpose behind this guided path was, as explained in chapter 3, that it should be possible for the pupils from grade one to grade nine to 'enter' the task, and start working on it. The intention was that the threshold 'into the task' should be so low that the pupils more or less at once should be able to produce or generate data that could act as a basis for an exploration that concluded in statement of relationships or rules.

How to organise or systematise these data was, however, an open question; i.e. the pupils were in charge for the organisation or systematisation of the data. For most of the solutions this organising was done as tables. For some of the tasks the pupils were also challenged to give a proof or argue for the correctness of the stated relationship(s) or rule(s).

As could be expected the analysis confirmed that the pupils in greater or less degree applied different solving procedures when they attacked the tasks. The analysis of the solutions revealed also that the solving procedures relied heavily on some fundamental mathematical structures. The analysis uncovered that each of the answers stated one or more relationships or rules. Some of these statements have an explicit structure, which will be called explicit statements, while others have a recursive structure and will be called recursive statements. Looking at all the analysed solutions, the number of explicit statements is higher than the number of recursive statements. Explicit statements are found in nearly all the solutions. Recursive statements are however not presented occasionally they are found in the solutions to four of the six tasks. Thirteen out of 23 solutions presented recursive statements, see table 6.2 (page 122), table 7.1 (page 142), table 8.2 (page 176) and table 10.2 (page 217). Mathematically, the stated relationships or rules can be identified as functions, in some instances functions with more than one independent variable. All the classes used functions in their solving procedures.

The mathematical structures inherent in the solution procedures can be localised at two levels, a top level and an underlying level. The top level embraces the mathematical structures that can be identified in the relationships or rules that the pupils clearly have stated in their answers. At this level the mathematical structures are clearly visible. The underlying level consists of mathematical structures that are applied in the solving procedures but not emphasised or (directly) focused by the pupils in their stated relationships or rules. The mathematical structures found in the underlying level are to a certain extent hidden or concealed.

The function structure is visible at the top level, but it is also present in the underlying level. At the top level functions are identified directly as tables or through the rules or relationships stated as a formula presented in a mathematical symbolic language or in the Norwegian (Swedish) language. The tables found in the solutions can in most cases be regarded as function tables. As pointed out elsewhere tables were present in a majority of the solutions. The extent of the tables varies from relatively few function values to many values. Class E's function table (page 106) contains six values while Class V's has $40 \times 8 = 320$ values (Class V, p. 4). See also the solutions to Class A (page 79), Class C (page 97), Class D (page 102), Class J (Class J, p. 1-2), Class M (page 150) and Class T (page 198).

The function structure at the underlying level can be identified (indirectly) through the solving procedure. One example of such identification is in the solving procedure applied by Class D (page 104), another is in the solution to Class N (page 152). In this particular case it is argued that the pupils in their solving procedure arrived at the stated rule via two other functions, which in this case meant that they applied composite functions. Another example of applying procedures at the underlying level was demonstrated by Class F in their solving procedure for the number of twin towers (page 111). In this case it is argued that they applied two different recursive rules that mathematically not can be identified as functions since the rules not are unique. A starting value was given and was used as input to the two recursive defined rules and the outcome(s) was then, if possible, used as input to the rules and so on.

Another underlying mathematical structure that has been identified in some of the solution procedures is equivalence relations. This mathematical structure appears in the following way; the data produced by the pupils, which for these tasks were sets of numbers, was separated into disjoint subsets where the elements in each of the subsets had an identical property. This means that the subsets can be identified as equivalence classes, and based on the study or inspection of these equivalence classes, a relationship or a rule was conjectured, see for example Class A (page 80). In some cases the equivalence relation was stated explicitly, as was the case for example Class B (page 88), Class D (page 104), Class H (page 115 and 117) and Class I (page 119).

A third mathematical structure identified in the solution procedures is the use of figurative numbers; odd/even, triangular numbers and square numbers. This mathematical structure is found in all the solutions to the tasks *A set with ten cubes* (chapter 6) and *Black and white squares* (chapter 8). Strict mathematically the idea of odd/even belongs to the equivalence relations, while the triangular and square numbers can be located to the function structure.

In the delimitation of the study, see section 1.6, it is emphasised that mathematical structures as natural numbers, order, and the four arithmetic operations not will be focused in this study. It is however tempting to mention that both Class A (page 82) and Class J (page 133) in their argumentation for the correctness of the stated rule explicit referred to important qualities of the algebraic structure called a ring.

For most of the classes the pupils assumed a stated relationship or rule to be true if they had tested it correct for some examples, see for example Class B (page 91), Class D (page 107), Class L (page 139) and Class W (page 216). In some instances the pupils emphasised that they tested or controlled their conjecture applying an alternative procedure, for example Class O (page 157) and Class T (page 198). In some cases the pupils argued or even gave a formal prove for the stated conjecture, see for example Class A (page 82), Class C (page 99), Class I (page 120), Class J (page 132) and Class R (page 174). The geometrical pattern explored in the task *Black and white squares* was for all of the Classes N to R observed to be two quadratic patterns interwoven in each other. This observation implied that they most likely considered it unnecessary to stress additional arguments for the correctness to the stated rules. However, it is appropriate to remark that not all the tasks encouraged or challenged the pupils to prove or argue for the correctness of their stated relationships or rules. That proofs just occurred in some of the solutions does therefore not necessarily mean that the pupils not were able to carry out a proof. It is noticed that it was not only the oldest pupils that proved or argued for the correctness for the statements, also classes at lower grades did this, for example Class C a grade 1 class, and Class J, a grade 3 class.

As has been explained in chapter 4 the answers to the tasks have not been prepeared in a vacuum, the solving process took place in a context. A crucial question is then which factors influenced the solving process and hence the solving procedures? If these factors can be identified to what extent did they influence on the solutions procedures and hence the findings? These questions will be addressed in the next section.

11.2 Factors influencing the solving procedures

The way the tasks were formulated is one factor that could have influenced the choice and implementation of the solving procedures. Another factor is the mathematical skills or the mathematical maturity, to the pupils. A third factor is the teacher, and a fourth is the competition rules. The main question is if it is possible to determine the degree of the influence to these factors. This question has to be raised separately for the different factors, which implies that several sub-questions had to be addressed.

1. To what extend guided or pointed the formulation of the tasks to a particular solving procedure?
2. What about the impact of the pupils' mathematical maturity for the choice and implementation of the solving procedure?
3. What was the teachers' influence on the choice and implementation of the solving procedures?
4. What about the impact of the competition rules on the solving procedure?

However, it is not claimed that these four factors are the only ones that influenced the pupils' solving procedures. There are certainly other factors that are of importance for the outcome of the solving process for example factors linked to the class milieu, the milieu surrounding the classroom and the available resources. One factor in the context of the class milieu is the collaboration aspect and how trained the pupils are to co-operate. For the milieu surrounding the classroom possible factors are the parents, brothers and sisters and friends. Since the data corpus for this study not gives or give very little information about these factors their influence or degree of influence will not be discussed. An important factor linked to the resources is time; how many lessons could be used on the solving process? There are some data linked to this factor, this will be touched upon in the question dealing with the teachers influence. The next sections will discuss the four questions raised above.

11.2.1 The formulation of the tasks

The objective of the tasks was to look for patterns then state or conjecture relationships or rules and for one half of the tasks also to argue or prove the correctness of the stated relationships or rules. The ‘guided path’-structure of the task encouraged the conjecturing to be based on data, which was generated through experimentation. For some of the tasks it was emphasised that the data should be organised in a practical way before they were inspected or studied. The tasks invited therefore the pupils apply a particular working method when they solved the tasks. This working method can be separated into different phases, which can be characterised:

1. Experimenting
2. Systematisation
3. Inspection
4. Conjecturing
5. Control
6. Proving

The first five of these phases can be identified with the working method which in the literature is called the *inductive approach* or the *discovery method*, the sixth phase presupposes deductive reasoning and belongs therefore to the *deductive method* (Christiansen, 1969). To what extend did the formulation of the tasks guide the pupils to the mathematical structures inherent in their solving procedures?

The experimentation phase invited the pupils to choose different numbers and carry through the prescribed activity on those numbers; mathematically to handle an independent variable. For half of the tasks a variable of dimension one, for the rest a variable of dimension greater than one; for example the variable in the task *The hundred square* (section 7.2.1) has dimension three, the variable in the task *In the city* (section 10.2) has dimension two, and for the task *Arithmogons* the dimension is four in the first part, and three the last part (section 9.2.3). The result of the activity is a dependent variable. The whole process, choosing an independent variable and calculating the dependent variable gives an element in a function table.

As mentioned above, the systematisation phase, phase 2, was not emphasised for all the tasks. However it appeared that all the classes by some means organised their data. Without regard to the practical organising of the data the end product of the experimentation and organisation phases can be characterised as a table, a function table.

The next phase, the inspection phase, which acts as a fundament for the conjecturing phase, can be executed in two fundamental and different ways; i.e. two different strategies can be applied. One strategy is to look for relationship(s) between the independent variable and the dependent variable; another strategy is to investigate for relationship between the dependent variables. A conjecture or relationship based on first of these methods will have an explicit structure if based on the second it will have a recursive structure. The formulation of the tasks did not specify in what way the inspection should be carried out. With one exception the tasks was constructed without any secret thought of what type of relationship the pupils would state; type here in the meaning of explicit or recursive. The task *In the city*, *Tangenten* 4-91, has, as will be remembered, Pascal's triangle as the underlying element, and as many mathematics educators have stressed, the Pascal triangle is mathematically very rich and therefore suitable for mathematical investigations. This was one important reason for constructing and using this task in the competition, but it was not the only reason. In the competition this task was scheduled after the task *Black and White Squares*, *Tangenten* 2-91. The evaluation, in the competition context, of the solution to Class M (see section 8.4) on the task *Black and White Squares* revealed, quite unexpectedly, that relationships presented by this class had a recursive structure. As a consequence of this observation the following question was posed: Was the formulation of a recursive structure by chance, or was it a structure that was familiar for the pupils?

However as the analysis has revealed, recursive statements are also present in solutions to the tasks *A set with ten cubes* and *The hundred square*, which in the competition came ahead of the task *Black and White Squares*. In this context is it therefore essential to mention that at the time it was observed that Class M had stated recursive relationships, the analysis of the solutions to the previous tasks had been carried out relatively superficial, and as a consequence the recursive statements in these solutions had not been observed or un-earthed. In order to try to find an answer, or approaching an answer to the above question, it was decided to construct a task where a recursive structure was fairly visible, which meant that it was most likely that the pupils would make statements focusing on this structure. The outcome of this deliberation was the task *In the city* where Pascal's triangle is the number pattern to explore. As is well known the recursive structure that can define Pascal's triangle is relatively visible or traceable in the number pattern, which is not the case for the explicit formula that can define this number pattern. The solutions to this task revealed that three out of four solutions discovered and presented the recursive structure. However, one pupil, a boy, in Class V discovered by chance, according to the teacher of the class (Class V, p.1), that the problem was identical or analogue to the calculation of *Lotto* numbers, and then he knew or identified the explicit formula. The only class participating with a solution to this task which did not state the recursive relationship was Class W. As has been pointed out, section 10.7, this class presented an incorrect explicit relationship. In their solution there was no trace to a recursive relationship. The question is why this class overlooked a relatively visible recursive relationship, and focused only on an explicit relationship. In this respect it is noticed that Class W was a group of pupils at grade 9, at that time, 1992, the top grade in the lower secondary school in Norway, which had chosen mathematics as an optional subject. Compared with the other solutions to this task the solution to Class W is a short one. There is for example no table or sutvey of the number of shortest ways, they answered only the first two questions the number of ways from *A* to *B* and from *A* to *C*. Looking at their solution it seems that their aim was to construct an explicit formula that produced the number of shortest ways to an arbitrary position. One wonders if the pupils have the opinion; in order to be characterised as a mathematical relationship the relationship has to be an explicit one. This question was also raised on account of what Class K remarked in their solution, page 136.

The task *In the city* was, as stressed above, the only task in the competition that was constructed with the purpose (secret thought) of challenging the pupils to look for recursive relationships. Looking at the other tasks and the solutions to those tasks it is noticed that most of the recursive relationships are stated by the classes in the primary school. Can this observation indicate that in the mathematics lessons in the schools or the teaching of the subject in the schools it is only the explicit relationships that are focused that a mathematical relationship or a formula, *per se*, is an explicit relationship/formula?

The formulation of the tasks invited the pupils to apply a particular working method, the discovery method. Based on the produced data (numbers) the pupils should look for patterns or relationships, this guided to the function structure, out over that the formulation was open it was not emphasised that one should look for relationships between this and that. Except for the function structure, it can therefore be concluded that the formulation of the tasks did not guide the pupils to the mathematical structures identified in their solving procedures. Another question is however how or in which manner the mathematical ideas or relationships behind the tasks directed the pupils to apply or identify particular mathematical structures; i.e. which mathematical structures are inherent in the mathematics exposed in the task?

11.2.2 Solving procedures in relation to mathematical maturity

The question concerning the impact of the pupils' mathematical maturity for the choice and implementation of the solving procedures is problematic. The main reason for this is due to the fact that very little is known about these pupils' mathematical background. Information touching on the mathematical maturity is only stated for two of the classes, see section 4.2.4. What is known is the grade of the class and therefore indirectly the age to the pupils. It will therefore be possible to compare the different solving procedures in relation to age.

The comparing of the solving procedures can be executed in two different ways. One way is to compare the classes' solving procedures on one particular task, another way is to compare across the tasks. Both will be discussed. The solving procedures for one particular task will be discussed first. Due to limited data basis for some of the tasks, one or two solutions only, comparing of the solving procedures for a particular task is not relevant for all the tasks.

The analysis has un-covered that some classes pursued or explored the problems more intimate than others. This was expected by reason of the open-endedness to the tasks. As a consequence the solving procedures varied. However, the analysis revealed that for some tasks the same solving procedure was used by classes from both the primary school and the lover secondary school. An example is the three classes, Class D (grade 3), Class H (grade 7-8) and Class I (grade 9) and their solving procedures to the task *A box with ten cubes*. This does not mean that the solutions were identical; the use of mathematical symbols and terms varies from solution to solution. Another example is the solving procedures applied by Class T (grade 3), Class U (grade 5) and Class V (grade 8-9) on the task *In the city*. However, in this case the three solving procedures are not quite identical. All three classes stated the same recursive relationship. In addition Class U stated relationships not stated by the other two classes, correspondingly for Class V. A third example is the solving procedures to the task *Black and white squares*. In this particular task one of the challenges was to determine in more than one way the number of squares in a geometrical pattern. All the solutions except that from Class M (grade 2) applied the same solving procedure as one of the procedures, which means that also in this case classes in primary school and classes in lover secondary school more or less applied the same solving procedures. The above indicates that it is the task it self that guides to a particular solving procedure; i.e. that the solving procedure is inherent in the task.

What then comparing the solving procedures across the tasks? This question has to a large extend been discussed in section 11.1.1. All the classes applied functions in their solving procedure. As could be expected and also observed, the handling of mathematical symbols and terms improved with age. For example both Class I (grade 9) and Class R (grade 8-9) in their solutions handled the mathematical symbols and terms in a convincing way. From what is also remarked in section 11.1.1 the recursive statements is frequently conjectured by the classes of the lowest grades, while the majority of the explicit statements was stated by the classes of the highest grades. Can this observation be an indication that the pupils' socialisation into the mathematical world or sphere direct or indirect implies; in mathematics the explicit relationships that are the important ones? Support for such a hypothesis is found in the solution to Class K (grade 3) where they emphasise that they did not manage to find a rule, even though they had stated a recursive rule, see page 136. An indirect support for this conjecture is the solution to Class W (grade 9). This class stated an (incorrect) explicit relationship and overlooked the recursive relationship, see page 216. Included in some of the solving procedures were also equivalence classes, and the conjecture of the corresponding equivalence relation. These were identified in solutions given by classes in primary school as well as classes from lower secondary school, see section 11.1.1.

Looking at the solving procedures, and also the argumentation used in the proofs presented in some of the solutions, it is by no means evident that the quality to the argumentation is increasing with the grades; i.e. the higher grade the better or more convincing are the arguments. For example the argument to Class C (grade 1) for the impossibility of building two towers of equal height is equivalent with the corresponding argument to Class I, ninth grade. Another example is the proof carried through by Class A (grade 5) on the task *A number and its reverse*, and the argumentation to Class J (grade 3) on the task *The hundred square*. In both cases the classes presented proofs where the starting point was a fixed or particular calculation, but their argumentation had a general character. The number symbols used by Class A in their proof can be replaced by arbitrary symbols like *a*, and *b*, and the same arguments can be used on these symbols. However it is also appropriate to point at the proof demonstrated by Class I, see page 120. Taken into consideration the age of the pupils this is a remarkable proof. It reveals a mature mathematical reasoning. Nearly 30 years experience in teacher education has told this researcher that reasoning like the one found in Class I's proof, would be difficult to understand by a majority of the teacher students. A natural question to pose, but impossible to answer at this stage: Did a pupil that had an especial interest in mathematics carry through this particular proof?

11.2.3 The teachers influence on the choice and implementation of the solving procedure

The question of the teachers influence on the choice and implementation has been discussed thorough in section 4.2, and the conclusion is that it is difficult to state or conclude about this influence. This does not necessarily mean that they had a great influence on the solutions and the solving procedures used by the pupils. For some of the solutions one can however trace the teacher's influence more than in others. This is particular the case for the teacher to Class M and Class T, which was the same class. In the experimenting and the systematisation phase the teacher for this class was relatively active. He organised, conducted is probably a bit strong, the experiments and also the systematisation of the data. For the systematisation he constructed, for both tasks, the tables in which the pupils put the numbers worked out. It is evident that this organising by the teacher facilitated the pupils' 'entrance' into the task, and that they quicker could enter the solving procedure. However, it can not be concluded that the solving of the tasks in these particular cases solely rested on the organising to the teacher. The organising of the activity to Class T, when they investigated the probability question in the task *In the city* (chapter 10) was initiated by the teacher. It seems reasonable to suppose that this organising was essential for the outcome of the investigation, not in the respect that the teacher directly influenced the statements, but it is found very unlikely that the pupils could have started to attack this problem without any organising assistance from the teacher.

One consequence of solving open-ended tasks is that the solving process could last for a relatively long period of time. It is therefore reasonable to expect that the result of the solving process also is dependant on the time used on solving the task. Since the teacher is responsible for organising the subject at the class level it has as a consequence that (s)he also will influence the solving process through the time which the class is allowed to spend on the working with the task. This is an indirect influence on the solving procedure and it will also have impact on the mathematical structures applied by the pupils' in their solution procedures. One example that supports this supposition is the solution to Class T on the task *In the city*. This solution was worked out over a period of four weeks, see section 10.4 page 195, and the result was a very extensive product. However, according to the available data no definite conclusion can be drawn concerning the time aspect and the influence on the solutions out over the obvious one that the pupils must have time to complete the solution procedure.

11.2.4 The impact of the competition rules on the solving procedure

The competition rules required that the solutions should be worked out collaboratively; i.e. in a co-operative setting. In addition these rules also regulated and put restrictions on the teacher's behaviour during the solving process. That is discussed in the preceding section, and will not be discussed further in this section.

As has been explained in chapter 3.2, collaboration between the pupils was one of the objective to the competition. Some of the teachers reported that the collaboration aspect was one important reason for participating in the competition, see section 4.2.3. Letters attached to some of the solutions (Class L, p. 1; Class M, p. 1; Class P, p. 1; Class T, p. 1; Class V, p. 1) emphasised that the pupils worked collaboratively in groups. Descriptions in some of the letters inform that the collaboration had a practical aspect as well as a mathematical aspect (Class M, p.1; Class T, p. 1; Class V, p. 1). It was also reported (Class B, p. 1) that one of the relationships stated by the class was discovered by a pupil that was considered (regarded) to be 'weak' in mathematics. But the collaborative working environment had produced the data, numbers, from which this particular pupil was able to extract the relationship.

One premise for creating a collaborative working environment is communication between the involved. The significance and importance of a communicative milieu in mathematics education has been generally discussed in chapter 3.2, and will not be further addressed. However, related to this particular study the question about the communicative milieu in the classes and its significance for the solving process should be addressed. The available data for this study give, with a few exceptions, no information about the communicative milieu when the pupils worked on the tasks. Two of these exceptions are the Classes M and T, which was the same class. The first of these two exceptions is a letter where the teacher described how the class prepeared the solution the task *Black and White Squares* (Class M, p. 1). The second example is found in the solution to the task *In the City*. The working process is described relatively detailed by the teacher (Class T, p. 1), and by the pupils (Class T, p. 10-15). From both descriptions it appears that communication between the pupils during the solving process played a prominent role. Another example is indirectly reported in a letter from the teacher to Class B (Class B, p. 1).

There can be no doubt that for several of the classes collaboration between the pupils was an important factor for the quality of the solutions, and that in these cases the collaboration was a decisive factor for the preparation of the comprehensive solutions produced. However, based on the present data it is not possible to draw a more accurate or exact conclusion concerning the impact of the collaboratively milieu.

11.2.5 The influences on the solving procedures

The main question raised in section 11.2 was the degree of the influence to the factors affecting the solving procedures. This question was split into four sub-questions; each question was dealing with one of the listed factors. In the four preceding sections these four questions were addressed separately, the mutual importance of the factors was not discussed.

Based on the argumentation from section 11.2.1 it can be concluded that the formulation of the tasks is the factor with least influence on the solving procedures. However, the construction of a task is not done in a vacuum; it rests on one or more mathematical ideas. These ideas will act as a source for stating relationships. The discussion in section 11.2.1 and section 11.2.2 reveals that the mathematical structures identified in the pupils' solutions for some of the tasks, *A box with ten cubes*, *Black and white squares* and *In the city*, to a high degree are the same regardless the grade of the class (see page 225). Based on this observation it can not be concluded that the grade of the class is of little importance for the outcome of the solving process and hence the mathematical structures applied by the pupils, but there are indications that for some tasks this could be the case. If so, the mathematical idea underlying the construction of the task is important, and this idea has therefore a substantial influence on the mathematical structures applied by the pupils in their solving procedures.

As has been emphasised in section 11.2.2 the question of impact of the pupils' mathematical maturity on the choice and implementation of the solving procedure is problematic. There are indications that the mathematical maturity, related to the mathematical structures applied in the solving procedures to the classes, not necessarily increases with the grades. However, as could be expected, there is example that the mathematical maturity to one particular pupil, see page 223, has had decisive effect on the outcome of the solving process, but it is also reported from the teacher to Class B (p. 1) that it was a pupil which was perceived to be weak in mathematics that discovered and stated a particular relationship.

Even though the competition rules put relatively strong restrictions on the role the teacher could play in the solving process, the teacher's influence is noticeable especially for the lowest grades, see section 4.2.2. As is emphasised in section 4.4.1 with one or possible two exceptions it is the pupils that have written the solutions, in the exception cases the teacher has acted as a secretary and reported the solutions. There is therefore no reason to suppose that the teacher had an active role in the solution process in the meaning of guiding the pupils through the process and direct point at what to do. The influence of the teacher is indirectly via the way the working process was organised.

From the preceding it can be concluded that two of the four factors discussed have a more prominent role related to the solution procedure applied by the pupils on the solutions analysed in this study. These factors are the teacher and the competition rules. The teacher's influence is indirectly as responsible for the organising of the solution process, the competition rules through the collaborative aspect. It seems that the formulation of the tasks are the factor that is the least important of the four factor discussed. As has been pointed out above in this context, the *Tangenten* competition, the influence of the factor linked to the mathematical maturity to the pupils is problematic. There are indications that this factor is decisive for the out come of the solving process, but there is also indication of the opposite, consequently it is questionable to draw any conclusion concerning the influence of this factor.

11.3 Findings research question 2: Mathematical archaeology

The second research question directed in this study is:

> *Is mathematical archaeology applied on pupils' written solutions of open-ended tasks a suitable tool for increasing the knowledge about pupils' mathematical activity and their mathematical thinking?*

As has been addressed elsewhere in this chapter, one objective for this study was to excavate and explicit-make the mathematics in pupils' solutions of some open-ended tasks. The analysis was carried out with a particular focus on mathematical structures. This process has in chapter 1 been characterised as mathematical archaeology, and it has been demonstrated in the chapters 5 to 10, all together on 25 written solutions. These chapters establish that a mathematical archaeology can be used or applied on pupils' solutions of open-ended tasks in order to bring to the light or make visible the mathematical structures inherent in the pupils' solutions. This is demonstrated for mathematical structures located at the top level as well as those mathematical structures located at the underlying level, see section 11.1.1. The explicit-making of the mathematics in the pupils' texts has as a consequence that it is easier to access the mathematics in those texts.

In the preceding chapters it has been demonstrated that mathematical archaeology can be a suitable tool for increasing the knowledge about pupils' mathematical activity and their mathematical thinking. This work has exemplified that explicit-making of the mathematics in the pupils' texts gives valuable information or knowledge about both their mathematical activity and their mathematical thinking.

Chapters 5 to 10 establish that in most cases the decoding of the pupils answers into a mathematical language, the decoding process, caused very little difficulties. However, there were instances were the decoding process caused difficulties, Class A (page 82), Class K (page 135 and Class P (page 166), and in a few cases it was found impossible to carry through the decoding process, Class H (page 117) and Class I (page 121). In the first case the un-earthing of the mathematical thinking to the pupils' was straightforward, straightforward in the sense that the result of the decoding process was unique; in the second case it was necessary to enter an emendation process in order to uncover the pupils' mathematical thinking.

The archaeology carried through in this study has revealed that the pupils in their collaborative solutions in some cases have used mathematical structures not taught in school. Most likely, the pupils will first be taught some of these mathematical structures in mathematics courses at university or college level. An example is equivalence relations.

A mathematical archaeology uncovers the past; it gives knowledge about the mathematics identified in works that have been finished. This knowledge can be interesting, but not necessarily useful. The usefulness of this knowledge is related to what extent it can act as a premise supplier for the future. The challenge is how this knowledge can be used in order to improve the mathematics teaching.

The explicit-making of the mathematics identified in pupils' mathematical texts has implications in relation to both a *local* and a g*lobal* perspective. The local perspective is related to the practice at the classroom level, and concerns the learning process in the individual classroom. It has implications for the teaching of the subject; i.e. how to plan and to carry through the teaching, and how to revise and adjust the teaching. This arena is the arena for mathematics teachers, but it is also a field where researchers in mathematics education should play important roles. These roles should be played in close collaboration with the mathematics teachers. The global perspective concerns the school at large, such as development of curricula, the design and development of suitable tasks and teaching elements, education of mathematics teachers, etc. This arena, the global level, is the field for the researcher in mathematics education.

To discus the roles played by the actors at these two levels would be interesting, but it is not the focus for this study, it will therefore be handed over to future studies. To predict the result or outcome to these studies is questionable, and will not be done. However, looking back at the some of the mathematical texts analysed in this study, it is realised that the pupils possesses a remarkable mathematical potential. Which at all should not be surprising, nearly a hundred years ago the Swedish pedagogue Anna Kruse wrote:

> Would it not be possible even here [in mathematics] to make an attempt to convert the children into independent investigators? Should one not replace the reproductive work with a more creative?

But how should one herewith do that? Certainly, we should let the child take our hand and guide us. And we should see how they should lead us to roads, where we never would dare to go, if they not have showed us, that they would be totally safe. (Kruse, 1910, p. 7)

APPENDIX A

12. COMPETITION RULES AND THEIR DEVELOPMENT

12.1 Introduction

The first number of *Tangenten* was published in September 1990. Professor Stieg Mellin-Olsen, the founder and first editor of the journal, proposed in the spring 1990 the first version of the rules. The first solutions revealed that it was necessary to amplify the competition rules. This lead to the first revision of the rules, later on it became necessary to carry through further revisions. These revisions were carried out in co-operation with different persons. The revision of the rules concerning the practical aspects was done by Stieg Mellin-Olsen, Marit Johnsen Høines and this researcher. The rules concerning the economy (i.e. the amount and the number of prizes) were done by Stieg Mellin-Olsen and Marit Johnsen Høines staff member of Caspar Publishing Firm (information given by M. Johnsen Høines). Stieg Mellin-Olsen and this researcher carried through the revision of the rules concerning the evaluation of the solutions.

12.2 The first version of the rules

When the competition started in September 1990, and as it appeared in the first two issues of *Tangenten*, **1**(1) and **1**(2), the competition rules were:

1. The competition is for classes, not for individuals.
2. It is competed in three groups, grade 1-3, grade 4-6, and grade 7-9.
3. The teacher can assist the pupils with the interpretation of the text of the task.

(*Tangenten* **1**(1), p.7; *Tangenten* **1**(2), p.31)

In addition there were rules concerning practical information such as time limits, how and where to send the solutions and about the prizes. It was also stressed that the competition was open for classes in all the Nordic countries. This information had the form:

6. The class must find a code word for the solution. Write this code word on a closed envelope, which contains the name and address of the class. Write also the grade of the class on the envelope.
7. Send the solution to Ole Einar Torkildsen, Volda College of Education, 6100 Volda within (date). He will, together with his students, evaluate the solutions.
8. Tangenten will purchase at most 3 solutions for 1000 NCR each. The solutions will be presented in number (stated).

9. The competition is open for classes in all of the Nordic countries.

(*Tangenten* **1**(1), p.7; *Tangenten* **1**(2), p.31)

In the original Norwegian text these rules were not numbered. The numbering given here is in accordance with numbering used in the subsequent issues of the journal.

12.3 Comments on the rules

The rules reflect both philosophical and practical aspects. The philosophical aspects are discussed in chapter 3.2. The practical aspects concerning the collection of data are discussed in chapter 4.2, the other aspects will be discussed below.

12.3.1 Rule 1: Co-operation between pupils

One of the main objectives to the competition was co-operation between the pupils in a mathematical context. This is discussed in chapter 3.2.

12.3.2 Rule 2: Competing in different groups

The dividing of the pupils into groups, based on grades, was done with respect to several factors. The pupils of the lowest grades, infant school, would probably felt it unfair if they had to compete on the same task with the pupils of the highest grades of the lower secondary school. Another factor was the number of participating classes in the competition. In advance it was impossible to predict this number; i.e. the number of solutions submitted. The intention was to send the first issue of the journal to every Norwegian primary and lower secondary school (information from Mellin-Olsen). To what degree this was done successfully is impossible to figure out, and even if a school received this first issue of the journal it does not mean that the mathematics teachers at the school received this issue. The name of the journal, *Tangenten,* does not uniquely associate to mathematics. In the Norwegian language the word *tangenten* has a double meaning, it could mean a tangent, but it could also denote *a key* (of a piano), and one has met teachers who have told that the first time they heard about *Tangenten,* they thought it was a music journal. On the background of the intention there was therefore a possibility that many classes could respond and participate in the competition. If so, the evaluation of the solutions and the decision of which classes who eventually should receive a prize was supposed to be easier if the classes had to compete in three different groups.

12.3.3 Rule 3: The role of the teacher

Another of the main objectives of the competition was to ‘listen to the pupils own mathematical voices’. A consequence of that objective was that the teachers were not permitted to intervene when the classes worked out the solution; i.e. they were not permitted to pose hints or questions in order to promote the pupils solution. The role played by the teacher should be rather passive, it was presupposed that the teacher only should present the task for the class and arrange the practical preparations necessary for the class's participation in the competition. The teacher was only permitted to assist the pupils with the interpretation of the text of the task. The influence of the teacher is discussed in section 4.2.2.

12.3.4 Rule 6: Code word on the solutions

The intention of this rule was to assure justice in the evaluation of the solutions. Contingent friendships or knowledge of particular classes or teachers should not have any influence on the evaluation of the solutions. The mathematical educational milieu in Norway was and is still a small milieu. When the rules were settled in May/June 1990, the evaluator had been working as a mathematics teacher educator for about 18 years, and had in that respect personal knowledge of relatively many Norwegian mathematics teachers. It was desirable that this knowledge should, as far as possible, be 'minimised' or preferably, be excluded. All of the classes did not keep strict to this rule, but in the actual cases the evaluator had neither knowledge about the teacher to the class nor about the school. The practising of this rule was not strictly enforced; solutions were not refused if they lacked the code word. The code word used could be relatively imaginative or descriptive; e.g. one class used 'Beagle Boys', another class used 'the wronghead'.

12.3.5 Rule 7: Evaluation of the solutions

This rule stated that the evaluation of the solutions should be carried out as a co-operation between this researcher and his students at the college. The rule did not explicitly state that it was the mathematics students who would be involved in this evaluation process. However, this was indirectly stated by the formulation 'his students' since the researcher only was teaching mathematics. The main reason for incorporating the students in the evaluation process was practical; they could facilitate the evaluation process. At the time the rules were settled the number of subscribers of the journal was unknown, and it was therefore impossible to predict the number of participating classes. As mentioned in paragraph 12.3.2 the intention was to send the first issue of the journal to every Norwegian primary and lower secondary school. If only a small percentage of the classes in the Norwegian primary and lover secondary schools responded and decided to participate in the competition, the evaluation of the solutions would be comprehensive, and would require more than one person. It appeared however that the percentage of Norwegian classes, which participated in the competition, proved to be what may be called, microscopically, and there was no need for assistance in the evaluation of the solutions, it was doable for one person. The students did never take part in the evaluation of the solutions.

There is also another argument for this rule that has to be taken into account, though with a certain degree of uncertainty. The uncertainty is due to the fact that this argument was, as far as remembered and in relation to rule seven, never discussed with Mellin-Olsen, it had however been raised and discussed with him in other contexts. The argument was the desirability for giving the mathematics teacher students an opportunity to ‘come close’ to the pupils. If the students were active in the evaluation process they would get a thorough knowledge of the pupils’ solutions, and hopefully then also to the pupils and their mathematical potential.

12.3.6 Rule 8: Prizes offered in the competition

It was competed in three groups, rule 2, and *Tangenten* would purchase at most three solutions; i.e. at most three prizes would be handed out. The rule did not state that it was at most one prize for each of the groups, but this was tacitly understood. It was nevertheless not the meaning that a single solution in one of the groups qualified for a prize. A possible prize depended on the quality of the solution. The rule was never enforced in that strict interpretation, at most one prize to each of the groups. More than one prize could be handed out in one of the groups, but the number of prizes did never exceed three.

12.3.7 Rule 9: A Nordic competition

This rule stated that classes in all the Nordic countries could participate.

12.3.8 The revision of the rules

In the light of the received solutions for the two first competitions, it turned out to be necessary to amplify and adjust the competition rules. The first revision of the rules occurred in *Tangenten* **2**(1), and the last one in *Tangenten* **2**(4).

12.4 The second version of the rules

The first revision of the rules was in February 1991. One cosmetic outcome of this revision was that the rules were numbered from 1 to 9 (*Tangenten* **2**(1), p.13).

The rules which were numbered 1 to 3 were identical with the three rules listed in the beginning of section 12.2. Those numbered 6 to 9 stated the practical information as done by the former rules with the same numbers (see section 12.2). Apart from some slight linguistic adjustments, the new rules 6, 8 and 9 were identical with the former rules 6, 8 and 9. Rule number 7 was changed to:

7. Send the solution to Ole Einar Torkildsen, Volda College of Education, 6100 Volda, within (date). (*Tangenten* **2**(1), p. 13)

This change just confirmed the introduced practice, see paragraph 12.3.5.

There were two new points or rules on the list, number 4 and 5, which was:

4. The evaluation of the solution is based on the pupils’ own work, and not on the teacher’s report of the pupils’ work.

5. One invites the classes to send in solutions even if they not have responded to all of the questions. (*Tangenten* **2**(1), p.13)

12.4.1 Rule 4: The pupils' own solutions

The intention with rule number 3, as presented in the first two issues of *Tangenten*, was to inform the teachers what they was allowed to do; i.e. that a teacher could only assist the pupils with the interpretation of the text of the task. It appeared, however, that one of the received solutions to the first task was a report and/or a summary of the pupils' work written by the teacher. This was not the intention since one of the main objectives of the competition was, as commented on in section 4.2.2, 'listen to' the pupils own voices, not the teachers interpretation of what (s)he observed. To avoid a similar situation in the future it was found necessary to emphasise rule number 3, and rule number 4 was therefore added.

12.4.2 Rule 5: Call on solutions

At the time this revision of the rules occured, only the two solutions of the task in issue **1**(1) had been submitted. The main reason for this rule 5 was to encourage more classes to participate in the competition, and sending in their solution even though they felt or meant that their solution was not perfect. To achieve this goal, to increase the number of solutions, it was found necessary to give a signal that a solution need not be perfect in order to participate in the competition. This was also stressed by Mellin-Olsen in a notice in *Tangenten* **2**(1) (p. 31) where gave a survey for the next issue of the journal.

12.4.3 Rule 7: Evaluation of the solutions

This version of the rule stated only when and where to send a solution. The explicit evaluation sentence of the former version was omitted. It was tacitly understood from the rule that the evaluation of the solutions would be done by the person mentioned.

12.5 The third version of the rules

In April 1991 the second revision of the rules occurred. This time the revision covered only rule number 4. The rule number 4 was now stated as:

4. The evaluation of the solution is based on the pupils' own work, and not on the teacher's report of the pupils' work. The teacher shall (neither) nor write the pupils' texts on a typewriter or in another way! (*Tangenten* **2**(2), p. 20)

12.5.1 Rule 4: The pupils' own solutions

Compared with the first version of this rule, an extra sentence had been added: "The teacher shall (neither) nor write the pupils texts on a typewriter or in another way!". The reason for adding this sentence was that one of the solutions to the task in issue **1**(2) was typed (with a typewriter), and the language used in this solution made it likely that the teacher had written the text (see sections 4.2.2 and 6.7). On the background of what was stressed in section 12.4.1, this was not acceptable, and it was found necessary to amplify the rule.

12.6 The fourth version of the rules

In September 1991 the next and third revision of the rules took place. This time the revision included the rules number 4, 8 and 9. These rules were now formulated:

4. The evaluation of the solution is based on the pupils' own work, and not on the teacher's report of the pupils' work. The teacher shall (neither) nor write the pupils' texts on a typewriter or in another way! It is for the benefit of possible reproducing in the journal that one writes on unlined paper sheets.

8. Tangenten will purchase at most 3 solutions for 500 - 1000 NCR each. To be taken into consideration for a prize the school had to subscribe to the journal.

9. Except for the restriction in rule number 8, the competition is open for classes in all the Nordic countries. (*Tangenten* **2**(3), p. 18)

12.6.1 Rule 4: The pupils' own solutions

This revision consisted in that one sentence that gave guideline for the type of paper the solutions should be written on, was added to the former version of the rule. The reason for this had its origin from the printing of the journal which also was stated the reason for. When a solution of a task was given in the journal, in order to illustrate this solution usually parts of the pupils' solutions would also be published. The direct cause for adding this sentence was the solution given by class C. This solution was written on lined paper and was difficult to copy. To get better quality of the copies of the pupils' solutions, and hence facilitate the printing of the journal, the classes were requested not to write their answers on lined sheets.

12.6.2 Rule 8: Prizes

Compared with the former version of this rule there are two changes. The amount of the prizes could be differentiated, and the school had to subscribe to the journal.

The main reason for the possibility of the evaluators to differentiate the amount of the prizes was to stress the importance of the quality of the solutions. The quality of a solution could be good, but not so excellent that they deserved a first prize. Another reason for this differentiation was for certain of economic nature. The number of subscribers to *Tangenten* was up till the time of this revision not high, and the saving of money was desirable. In fact the journal could not existed without economic support from Caspar Publishing Firm (information by M. Johnsen Høines, committee member of Caspar Publishing Firm).

The main reason for the subscription demand was to increase the number of subscribers, and hence of economic nature. Some of the classes participated in the competition even though the school did not subscribe to *Tangenten*. Their teacher subscribed private, but not the school. The intentions of including this restriction into the rule, was to get a 'light pressure' on the schools to subscribe to the journal.

12.6.3 Rule 9: A Nordic competition

The revision of this rule is consequences of the restriction put forward in rule number 8.

12.7 The fifth version of the rules

This revision was in November 1991. The foci were on rules number 8 and 9. This time they were formulated:

8. Tangenten will purchase at most 3 solutions for 500 - 1000 NCR each.

9. The competition is open for classes in all the Nordic countries. (*Tangenten* **2**(4), p. 19)

This was the last revision of these two rules.

12.7.1 Rule 8: Prizes offered in the competition

The former version of this rule had caused an intern discussion in Caspar Publishing Firm, the discussion was between Mellin-Olsen and Johnsen Høines. They did not agree on the former version of this rule (revision fourth), a version that put restrictions of who could participate in the competition. Johnsen Høines argued for removing the subscription demand, and as this revision shows that was the outcome of the discussion (information given by M. Johnsen Høines).

12.7.2 Rule 9: A Nordic competition

This version of rule number 9 was, as in section 12.6.3, a consequence of the revision of rule number 8.

12.8 The sixth and last revision of the rules

In February 1992 a minor change occurred in rule number 4. The sentence "It is for the benefit of possible reproducing in the journal that one writes on unlined paper sheets." was removed. The rule was now stated as:

8. The evaluation of the solution rests on the pupils' own work, and not on the teacher's report of the pupils' work. The teacher shall (neither) nor write the pupils' texts on a typewriter or in another way! (*Tangenten* **3**(1), p. 20)

This formulation is identical with the formulation found in the third version of the rules, see section 12.5.1.

For the remaining two competitions, those in issues **3**(2) and **3**(3), the rules were not changed.

12.9 Summary – Revisions of the rules

The following table summarises the revisions of the competition rules:

Tangenten	**1**(1) Sept 90	**1**(2) Nov 90	**2**(1) Feb 91	**2**(2) Apr 91	**2**(3) Sept 91	**2**(4) Nov 91	**3**(1) Feb 92	**3**(2) Apr 92	**3**(3) Sept 92
Revision			1st	2nd	3rd	4th	5th		
Rule 1	1								
Rule 2	1								
Rule 3	1								
Rule 4	ND	ND	1	2	3		4		
Rule 5	ND	ND	1						
Rule 6	1								
Rule 7	1		2						
Rule 8	1				2	3			
Rule 9	1				2	3			

Table 12.1 An overview of the revisions of the rules in the Tangenten competition

The numbers in the table means the version number of the actual rule. ND means that the rule was not defined.

From the above table it is observed that the rules numbered 1, 2, 3, 5 and 6 were not revised during the period of the competition. The rule number 4 had most revisions. All together there were four revisions of that rule, but since version number 2 and version number 4 is identical, there are three different versions.

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